Solution 3

Hand in Ex 2.4, no. 8, Supp. Ex (3) by Feb 7.

Ex 2.3

5. We cannot differentiate the equation $e^\theta = \sum_n c_n e^{in\theta}$, since the R.H.S. is not uniformly convergent.

7. (a) For $n \neq 0$,

$$|c_n| = \left|\frac{1}{n^{13.2} + 2n^6 - 1}\right| \leq \frac{1}{|n|^{13.2}} = |n|^{-(12+1.2)}.$$

It follows from Theorem 2.6 that $f \in C^{(12)}$.

(b) $b_n = 0$ for $n \geq 1$. For any fixed k, there is a constant C_k such that

$$|a_n| = \left|\frac{1}{2^n}\right| \leq \frac{C_k}{n^k} \qquad \text{for } n \geq 1,$$

since $\lim_{n\to\infty} \frac{n^k}{2^n} = 0$. It follows from Theorem 2.6 that $f \in C^{(k)}$ for any k. Hence f is infinitely differentiable.

(c) By Weierstrass M-test, f is continuous. Since

$$(2^m) a_{2^m} = \frac{2^m}{2^m} = 1 \qquad \text{for all } m,$$

it follows from Theorem 2.6 that f is not piecewise smooth.

Ex 2.4

1. $f(\theta) = 1$. The cosine series of f on $[0, \pi]$ is 1, which converges to 1 at both $\theta = 0$ and $\theta = \pi$; the sine series of f on $[0, \pi]$ is $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}$, which converges to 0 at both $\theta = 0$ and $\theta = \pi$.

3. $f(\theta) = \sin\theta$. The cosine series of f on $[0, \pi]$ is $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\theta}{4n^2 - 1}$, which converges to 0 at both $\theta = 0$ and $\theta = \pi$.; the sine series of f on $[0, \pi]$ is $\sin\theta$, which converges to 0 at both $\theta = 0$ and $\theta = \pi$.

8. Let $f(x) = 1 - x$ on $[0, 1]$.
   Set $\theta = \pi x$. From Example 1, we have

$$\frac{\theta}{\pi} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} \qquad \text{for } 0 < \theta < \pi.$$

Hence, for $0 < x < 1$,

$$1 - x = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{(2n-1)^2}.$$

9. Let $f(x) = 1$ for $0 < x < 2$, $f(x) = -1$ for $2 < x < 4$. Then

$$\frac{a_0}{2} = \frac{1}{4}\int_0^4 f(x)dx = 0,$$

and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^4 f(x)\cos\frac{n\pi x}{4}dx \\ &= \frac{1}{2}\int_0^2 \cos\frac{n\pi x}{4} - \frac{1}{2}\int_2^4 \cos\frac{n\pi x}{4} \\ &= \frac{2}{n\pi}\left(2\sin\frac{n\pi}{2}\right) \\ &= \begin{cases} \frac{4}{n\pi}(-1)^{\frac{n+1}{2}+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Hence

$$f(x) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}\cos\frac{(2m-1)\pi x}{4}.$$

Supplementary Exercise

1. Let

$$f^{(k)} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n^k\cos nx + b_n^k\sin nx).$$

Show that for an infinitely many times differentiable, 2π-periodic function f,

$$a_n^{2k} = (-1)^k n^{2k}a_n, k \geq 1, \quad a_n^{2k-1} = (-1)^{k+1}n^{2k-1}b_n, k \geq 0.$$

Then find a similar formula relating b_n^k to a_n, b_n.

Solution. By Theorem 2.2, for each k,

$$a_n^{k+1} = nb_n^k \quad \text{and} \quad b_n^{k+1} = -na_n^k.$$

Hence, for $k \geq 1$,

$$a_n^{2k} = -n^2 a_n^{2k-2} = \cdots = (-1)^k n^{2k}a_n$$

while

$$a_n^{2k-1} = -n^2 a_n^{2k-3} = \cdots = (-1)^{k-1}n^{2k-2}a_n^1 = (-1)^{k+1}n^{2k-1}b_n.$$

Similarly, for $k \geq 1$,

$$b_n^{2k} = -n^2 b_n^{2k-2} = \cdots = (-1)^k n^{2k}b_n$$

while

$$b_n^{2k-1} = -n^2 b_n^{2k-3} = \cdots = (-1)^{k-1}n^{2k-2}b_n^1 = (-1)^k n^{2k-1}a_n.$$

2. Verify that the following sequences are rapidly decreasing: (a) $x_n = 1/2^n$ and (b) $y_n = \sin n/n^n$. **Solution.**

(a) x_n is rapidly decreasing, since for each fixed k, $\lim_{n\to\infty} x_n n^k = \lim_{n\to\infty}\frac{n^k}{2^n} = 0$.

(b) y_n is rapidly decreasing, since for each fixed k,

$$|y_n|n^k = \left|\frac{\sin n}{n^n}\right| n^k \le \frac{1}{n^{n-k}} \le 1,$$

whenever $n \ge k$.

3. Show that the cosine series

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

is equal to a sine series on $(0, \pi)$. Find this sine series and check if these two series are equal at 0 and π.

Solution. Observe that the cosine series is the Fourier series of $f(x) := x^2$ on $(-\pi, \pi]$. The sine series of f on $(0, \pi)$ is given by $\sum_{n=1}^{\infty} b_n \sin nx$, where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx \\ &= \frac{2(-1)^{n+1}\pi}{n} + \frac{4}{n^3\pi}((-1)^n - 1). \end{aligned}$$

Thus, for $0 < x < \pi$,

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx = x^2 = \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}\pi}{n} + \frac{4}{n^3\pi}((-1)^n - 1)\right]\sin nx.$$

At $x = 0$, both series converge to 0; at $x = \pi$, the cosine series converges to π^2 while the sine series converges to 0.