Solution 1

Hand in no. 3 and 5 by Jan 24.

1. Prove the formula

$$\cos\theta + \cos 2\theta + \cdots + \cos N\theta = \frac{\sin(N+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{\theta}{2}}, \quad \theta \neq 0.$$

Solution By product to sum formula,

$$\cos n\theta \sin\frac{\theta}{2} = \frac{1}{2}\left[\sin(n+\frac{1}{2})\theta - \sin(n-\frac{1}{2})\theta\right].$$

Summing it over n, we have

$$\left(\sum_{n=1}^{N}\cos n\theta\right)\sin\frac{\theta}{2} = \frac{\sin(N+\frac{1}{2})\theta - \sin(1-\frac{1}{2})\theta}{2}.$$

Hence

$$\sum_{n=1}^{N}\cos n\theta = \frac{\sin(N+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{\theta}{2}}, \quad \theta \neq 0.$$

2. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series. A 2π-periodic function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for $x \in [-\pi, \pi]$.

Solution Write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Suppose $f(x)$ is an even function. Then, for $n \geq 1$, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0}\sin nx f(x)dx + \int_{0}^{\pi}\sin nx f(x)dx].$$

By a change of variable and using $f(-x) = f(x)$ since $f(x)$ is an even function,

$$\int_{-\pi}^{0}\sin nx f(x)dx = \int_{0}^{\pi}\sin(-nx)f(-x)dx = -\int_{0}^{\pi}\sin nx f(x)dx,$$

one has

$$b_n = \frac{1}{\pi}[-\int_{0}^{\pi}\sin nx f(x)dx + \int_{0}^{\pi}\sin nx f(x)dx] = 0.$$

Hence the Fourier series of every even function f is a cosine series.

Now suppose $f(x)$ is an odd function. Then, for $n \geq 1$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0}\cos nx f(x)dx + \int_{0}^{\pi}\cos nx f(x)dx].$$

By a change of variable and using $f(-x) = -f(x)$ since $f(x)$ is an odd function,

$$\int_{-\pi}^{0}\cos nx f(x)dx = \int_{0}^{\pi}\cos(-nx)f(-x)dx = -\int_{0}^{\pi}\cos nx f(x)dx,$$

one has

$$a_n = \frac{1}{\pi}[-\int_0^\pi \cos nx f(x)dx + \int_0^\pi \cos nx f(x)dx] = 0.$$

Furthermore, by a change of variable and using $f(-x) = -f(x)$,

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}[\int_{-\pi}^{0} f(x)dx + \int_0^\pi f(x)dx] \\ &= \frac{1}{2\pi}[-\int_0^\pi f(x)dx + \int_0^\pi f(x)dx] = 0.\end{aligned}$$

Hence the Fourier series of every odd function f is a sine series.

3. Show that the Fourier series of $f_1(x) = x^2, x \in [-\pi, \pi]$, is given by

$$f_1(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Solution As $f_1(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3},$$

and by integration by parts, for $n \geq 1$,

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi}x^2 \sin nx\Big|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi}x\cos nx\Big|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi} \cos nx dx \\ &= 4\frac{(-1)^n}{n^2}.\end{aligned}$$

4. Show that the Fourier series of $f_2(x) = -1, x \in [-\pi, 0)$ and $= 1, x \in [0, \pi]$, is given by

$$f_2(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

Solution As $f_2(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$. For $n \geq 1$,

$$\begin{aligned}b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^\pi \sin nx dx \\ &= -\frac{2}{n\pi}\cos nx\Big|_0^\pi \\ &= \frac{2[1-(-1)^n]}{n\pi} \\ &= \begin{cases} 0 & \text{if } n = 2m \\ \dfrac{4}{(2m-1)\pi} & \text{if } n = 2m-1. \end{cases}\end{aligned}$$

5. Show that the Fourier series of $f_3(x) = e^{bx}, x \in [-\pi, \pi]$, $b \in \mathbb{R}$, is given by

$$f_3(x) \sim \frac{\sinh b\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{b - in} e^{inx}.$$

Here the hyperbolic sine function is given by $\sinh \theta = (e^\theta - e^{-\theta})/2$.

Solution First we assume that $b \neq 0$. For $n \in \mathbb{Z}$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{b-inx} dx \\ &= \frac{1}{2\pi(b-in)} e^{b-inx} \Big|_{-\pi}^{\pi} \\ &= \frac{(-1)^n}{\pi(b-in)} \frac{e^{b\pi} - e^{-b\pi}}{2} \\ &= \frac{\sinh b\pi}{\pi} \frac{(-1)^n}{b-in}. \end{aligned}$$

If $b = 0$, then it is easy to see that $f_3(x) = 1 \sim 1$.