Origami and Mathematics

New Wave Mathematics 2018

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About Origami

Where did Origami come from?

50:50

A. China

B. Japan

C. Germany

D. Italy

China?

Reason: Paper invented in China around 105 A.D.

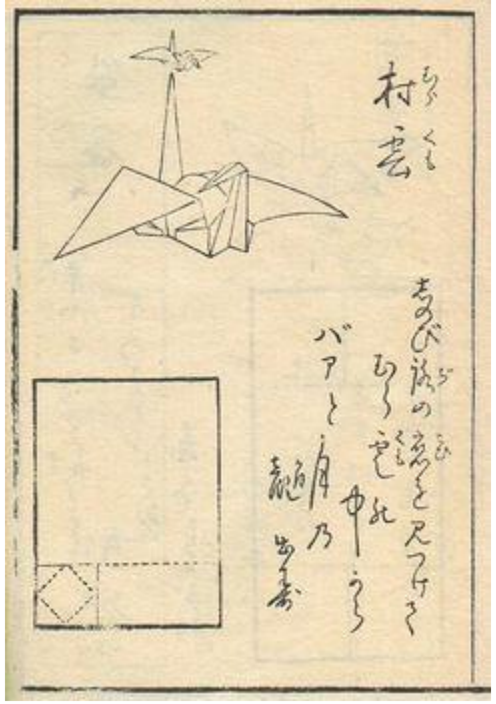
- use for padding and wrapping bronze mirror.
- use for writing by 400 A.D.
- use for toilet by 700 A.D.
- use for money by 960 A.D.

Sources: https://en.wikipedia.org/wiki/Bronze_mirror
https://ancientchina132.weebly.com/writing-material.html
http://blog.toiletpaperworld.com/toilet-paper-challenge/
http://www.thecurrencycollector.com/chinesebanknotes.html

Japan?

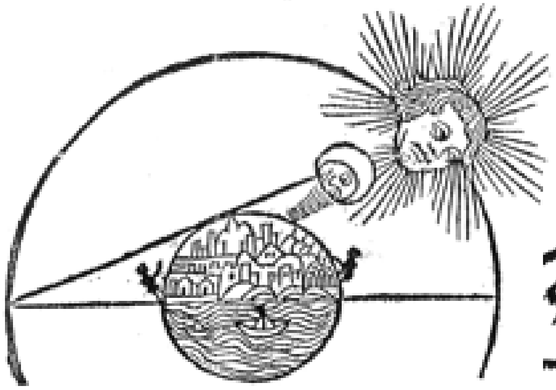
Reason: Origami comes from the words “ori” (to fold) and “kami” (paper).

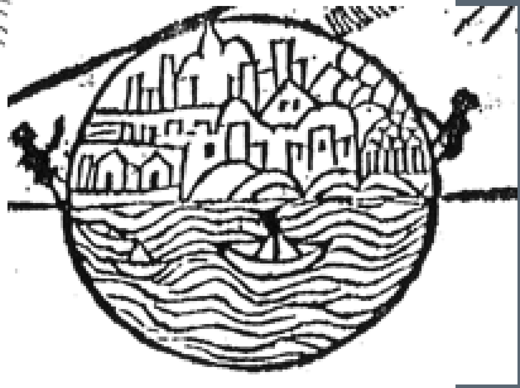
- Japanese monks used paper for ceremonial purposes around 600 A.D.
- Origami butterflies appear in Japanese Shinto weddings around 1600s.
- First origami book “Sembazuru Orikata” (Secret Techniques of Thousand Crane Folding) published in 1797.



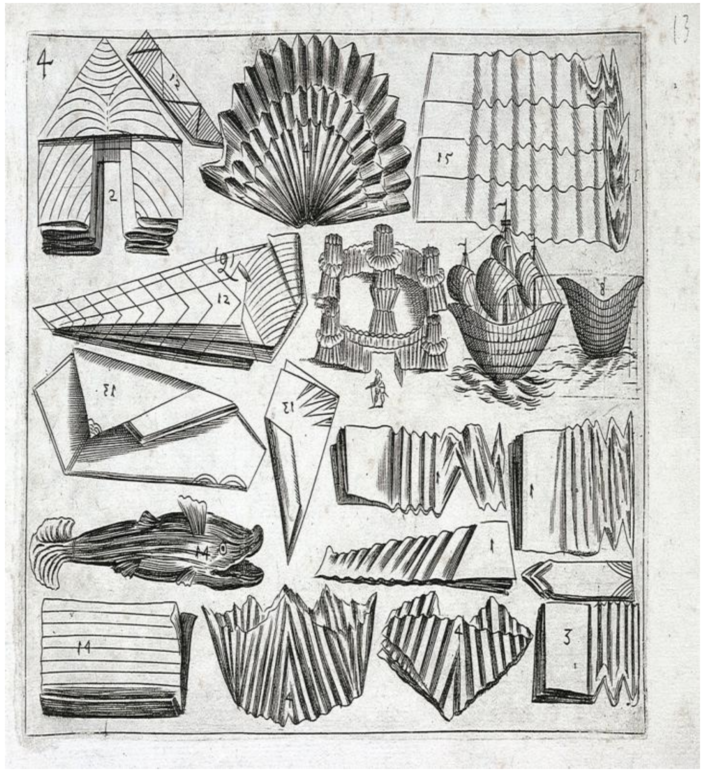
Sources: http://origami.ousaan.com/library/historye.html
https://en.wikipedia.org/wiki/History_of_origami
https://allabout-japan.com/en/article/4425/
https://www.origami-resource-center.com/butterfly.html

Europe?





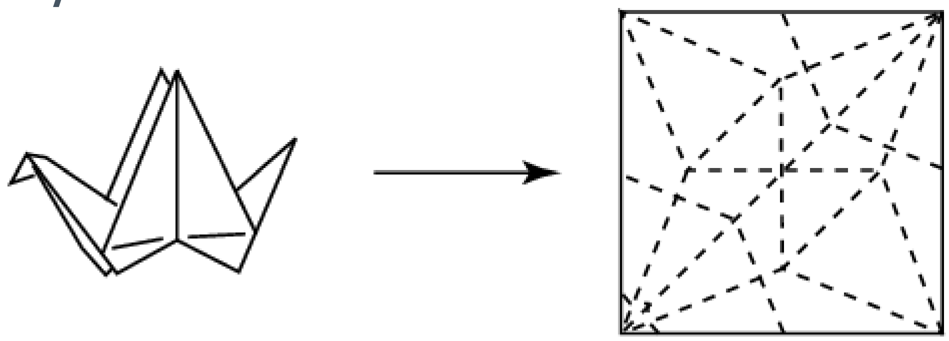
- picture of paper boat found in a medieval astronomy text *Tractatus de Sphera Mundi* published in 1490.
- references of “paper prison (紙籠)” in a English play (Duchess of Mafia) by John Webster, in 1623.
- an Italian book by Matthias Geiger, published in 1629, for folding table napkins (餐巾)

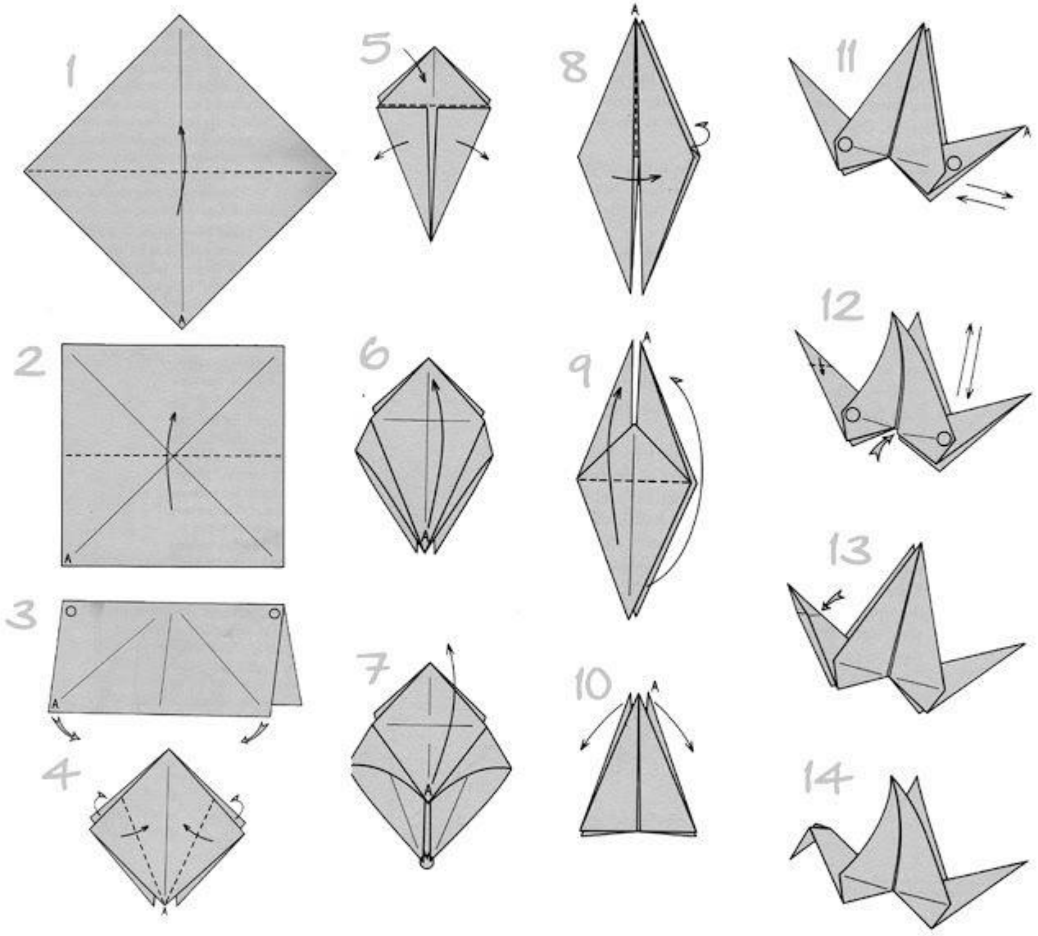


Sources: http://origami.ousaan.com/library/historye.html
https://en.wikipedia.org/wiki/History_of_origami
http://www.britishorigami.info/academic/lister/german.php
http://search.getty.edu/gri/records/griobject?objectid=242745552
http://www.florilegium.org/?http%3A//www.florilegium.org/files/CRAFTS/Paper-Folding-art.html

Modern Origami

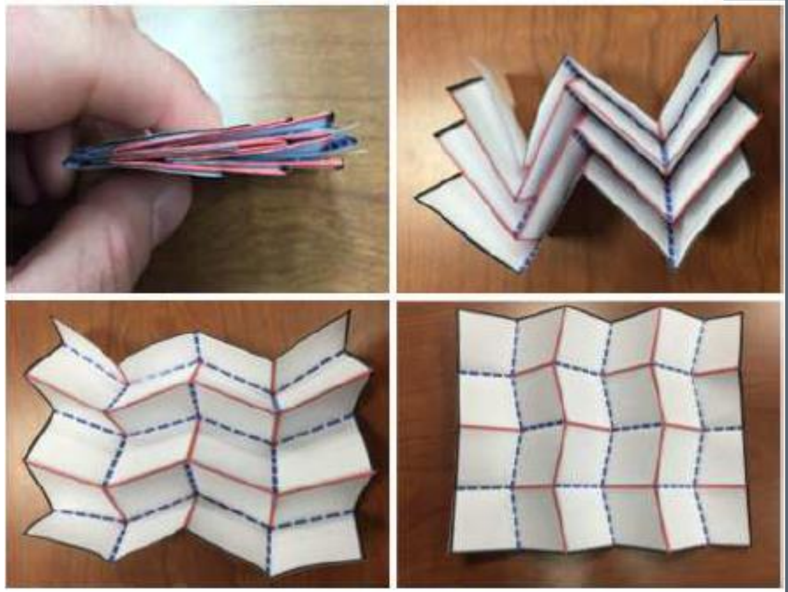
- Origami is now viewed as a puzzle – follow a sequence of folds to get a shape.
- Each origami can be unfolded to give a *crease pattern*.

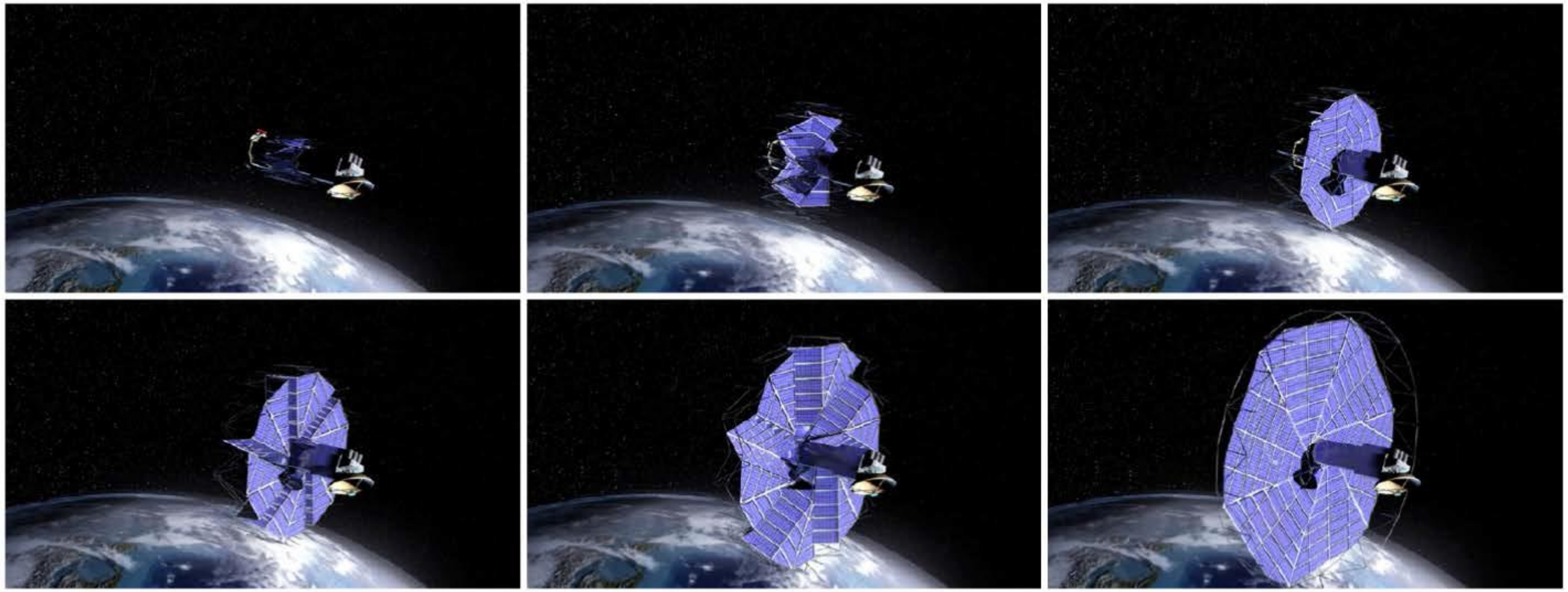




Source: https://i.pinimg.com/736x/e6/d9/30/e6d930629820c816130fa6d24939bdbc--origami-birds-origami-cranes.jpg
http://mathworld.wolfram.com/images/eps-gif/KawasakiCrane_1000.gif

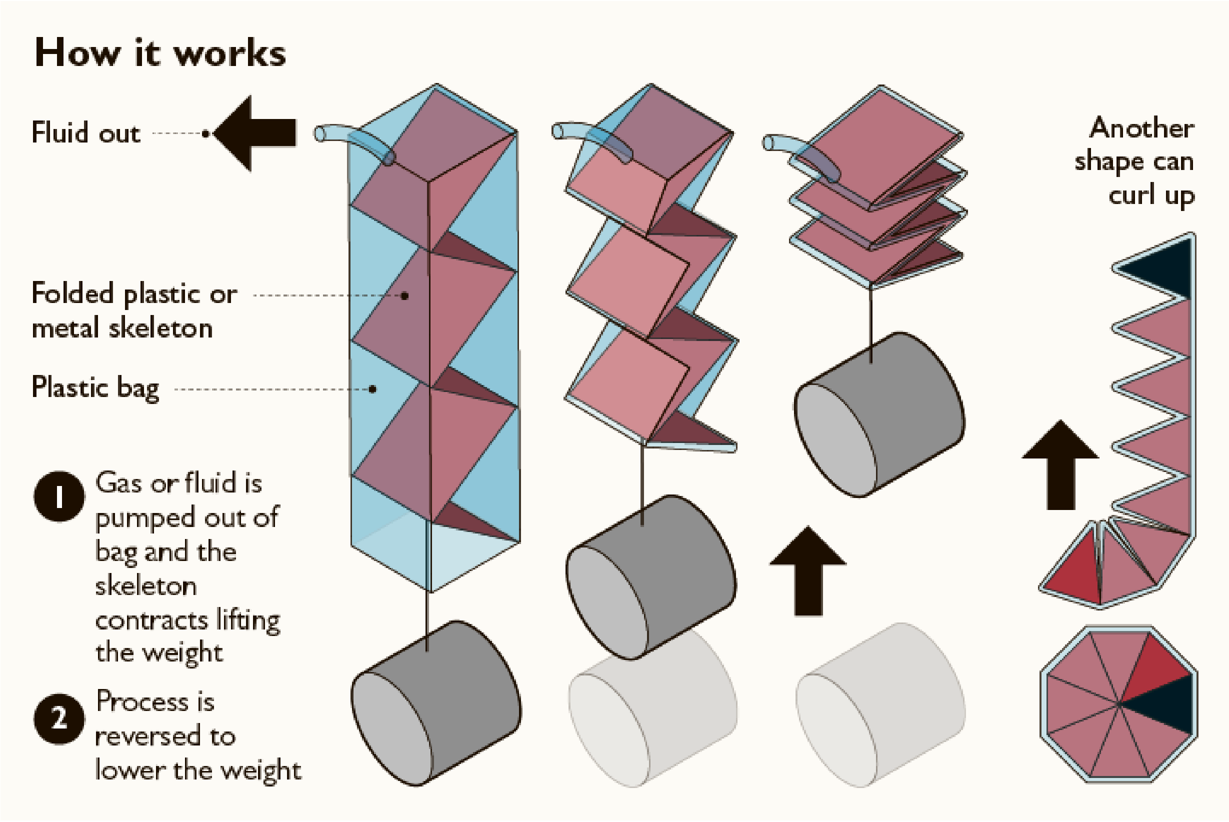
Miura-map fold for solar panels

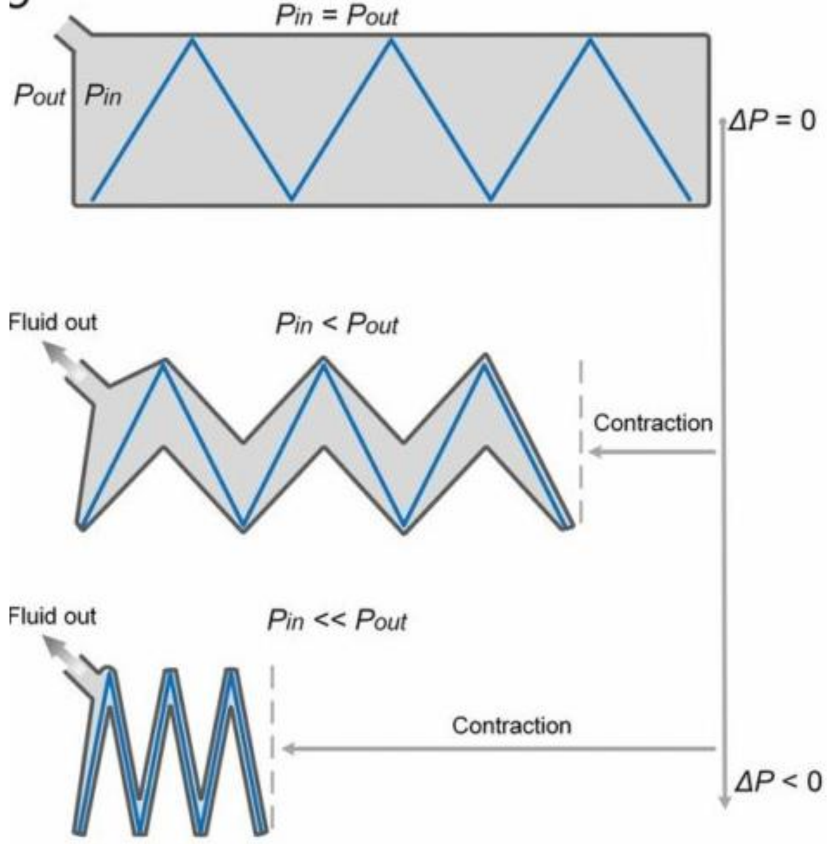


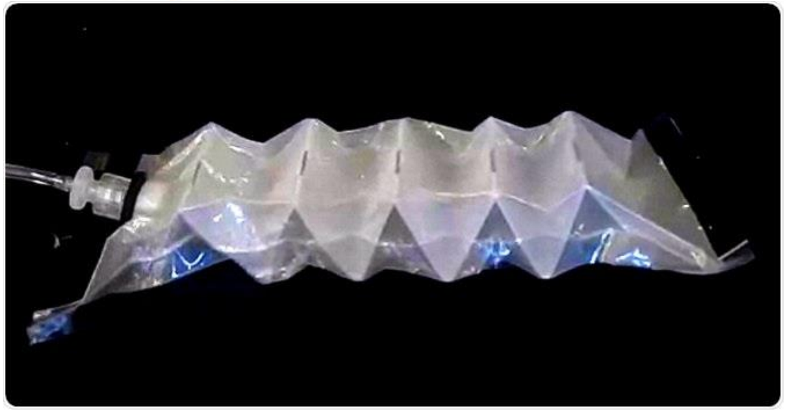


Source: https://engineering.nd.edu/spotlights/1BusEng1st20004000TurnerApplicationPackageComplete.pdf
https://www.comsol.com/blogs/solving-space-problem-origami-principles/

Origami muscles





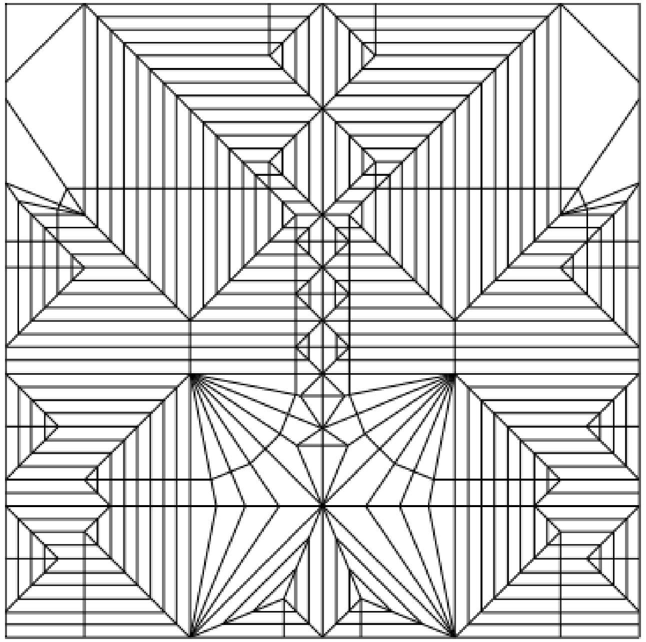


Source: https://www.zmescience.com/science/vacuum-fold-actuators-mit/
https://www.news-medical.net/news/20171127/Origami-inspired-artificial-muscles-can-lift-1000-times-their-weight.aspx
https://www.thetimes.co.uk/article/origami-inspired-muscles-bring-super-strong-robots-a-step-closer-j8b6k3899

The BIG question (Inverse problem 逆問題)

› Given an origami, find a crease pattern and the instructions that will lead to the origami.

?



› Some work on this (Treemaker – Robert J. Lang), (Origamizer - Tomohiro Tachi)

Source: http://jasonku.mit.edu/butterfly1.html

Origami constructions (摺紙建築)

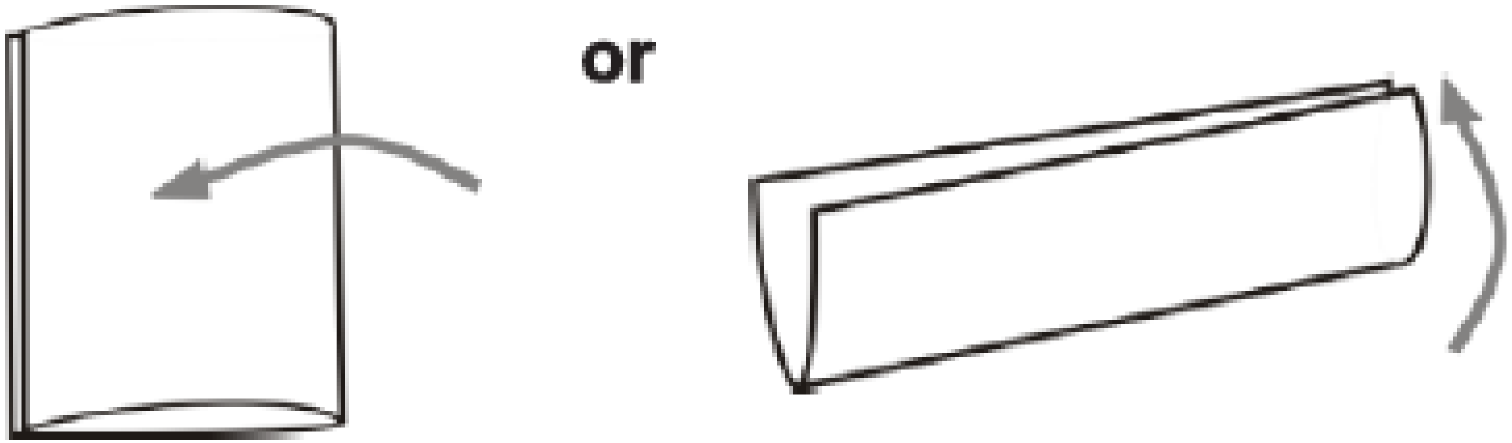
- During the folding process, you may have to create creases at specific points of the paper, e.g., 1/3th along the border.
- Within origami, there is an interest in creating these points just by folding.
- This has overlaps with the mathematical field of geometric constructions (幾何建築) e.g. bisect angles, finding the center of a circle, construction of triangles.

Source:

Origami for mathematics

Fujimoto's approximation (藤本估計)

- Dividing a piece of paper into $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ … is easy to do.



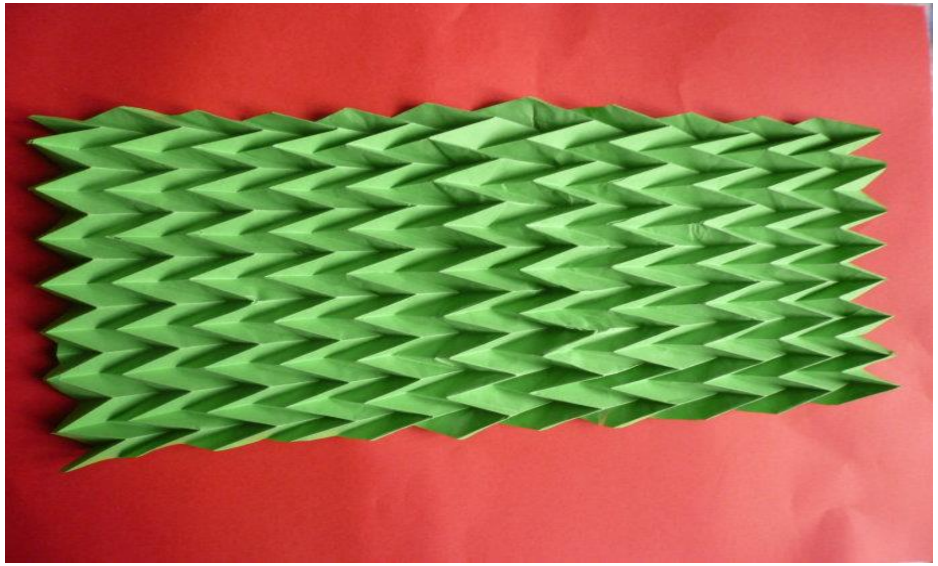
- But what if you want to divide into equal $\frac{1}{3}$ or $\frac{1}{5}$? E.g., fold a letter into an envelop?

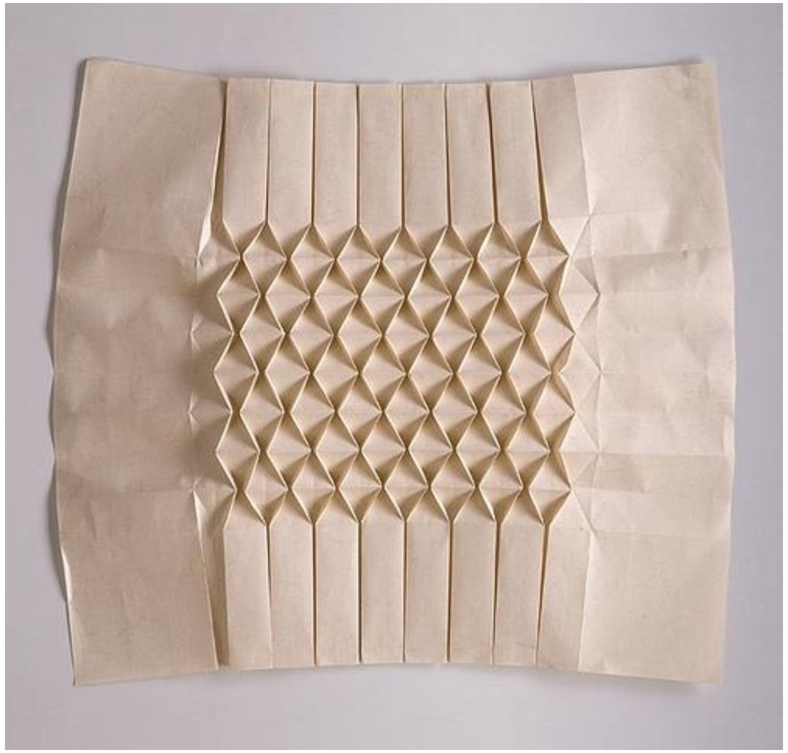
Fujimoto's approximation

› FUJIMOTO Shuzo (藤本修三)

- high school teacher in Japan;

- wrote a book <<编织折纸>> in 1976 and introduced how to fold patterns

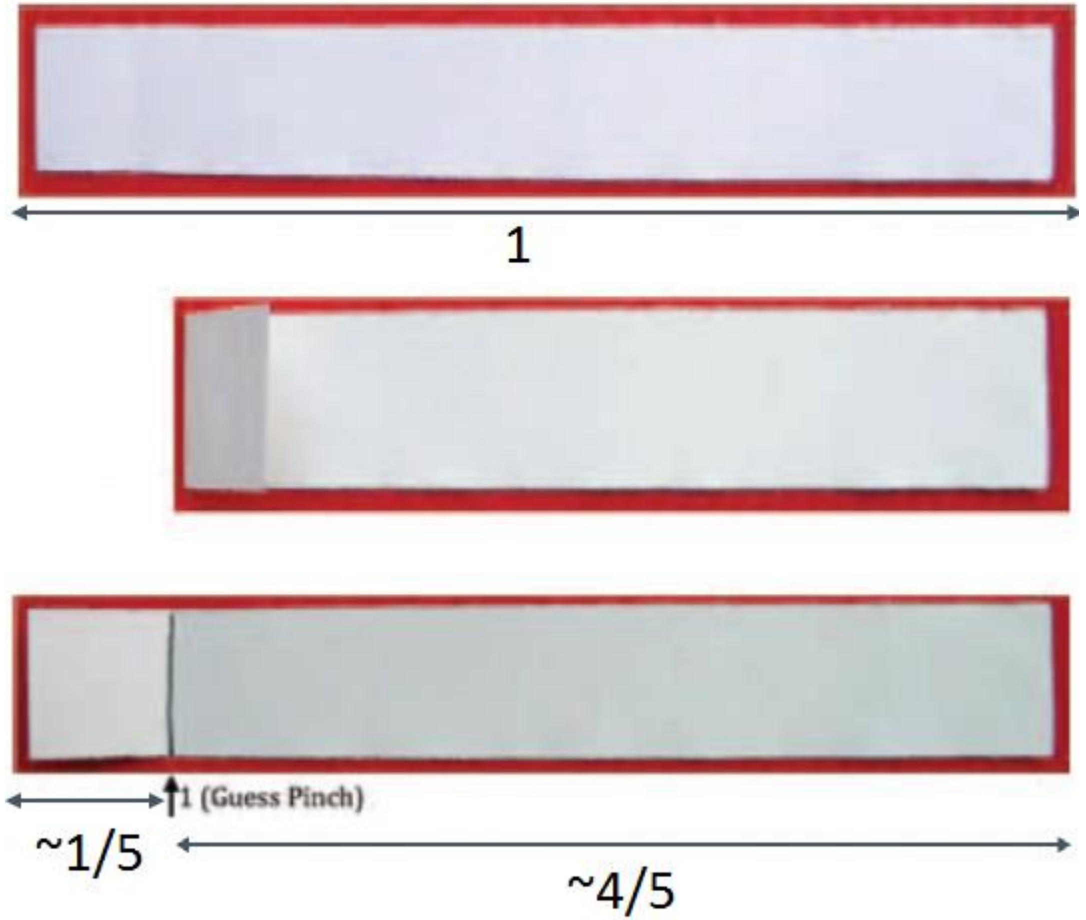




Source:
http://blog.sina.com.cn/s/blog_416862f50100mzyf.html
http://www.allthingspaper.net/

Fujimoto's approximation

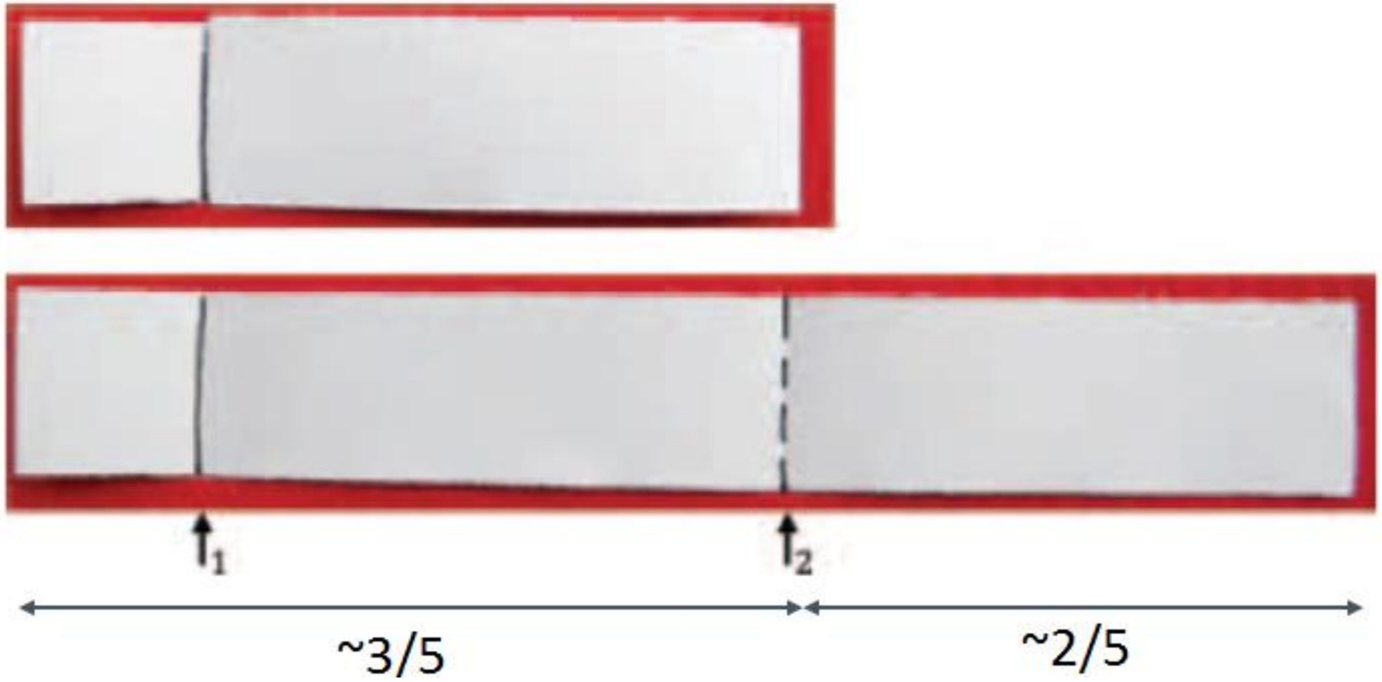
› Step 1: Make a guess where 1/5th mark is.



Source: http://www.teachersofindia.org/sites/default/files/5_how_do_you_divide_a_strip_into_equal_fifths.pdf

Fujimoto's approximation

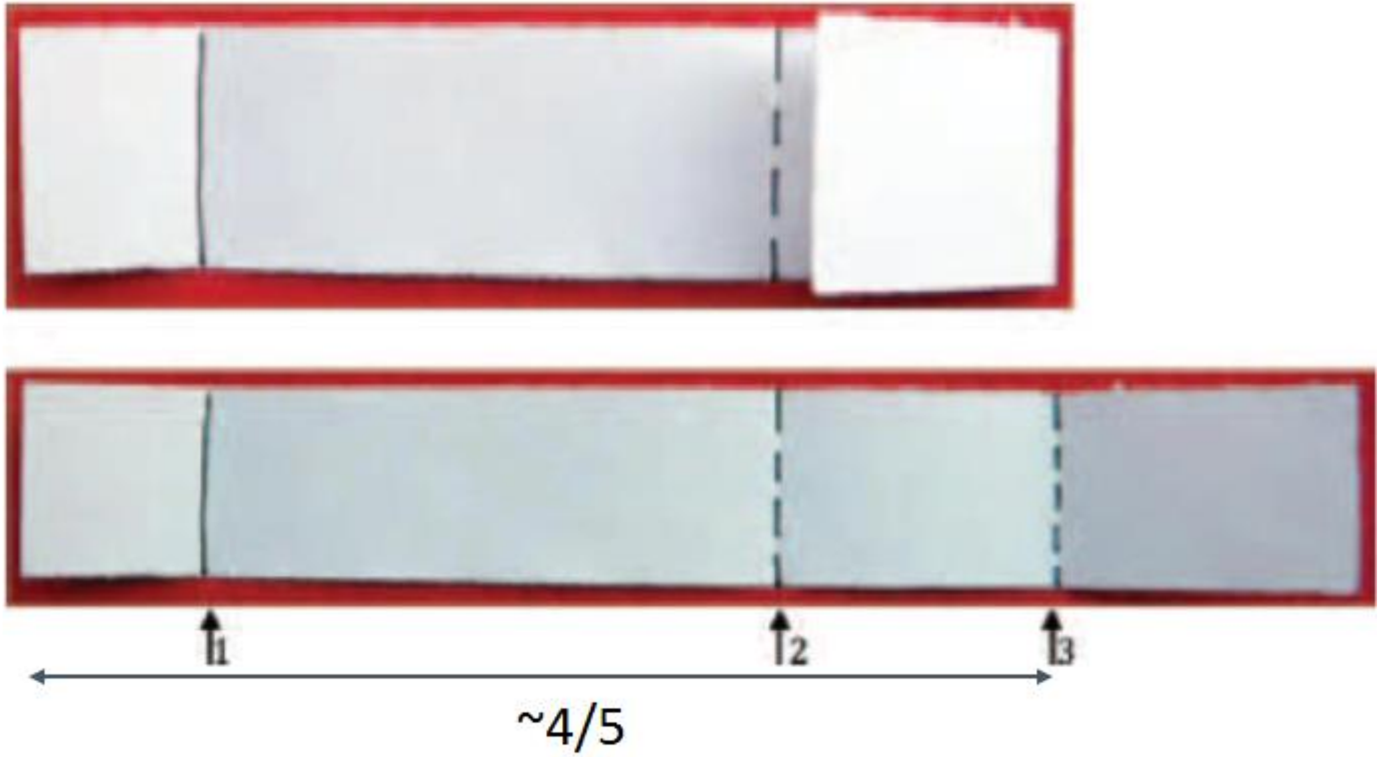
› Step 2: The right side is roughly 4/5 of the paper. Pinch this side in half.



Source: http://www.teachersofindia.org/sites/default/files/5_how_do_you_divide_a_strip_into_equal_fifths.pdf

Fujimoto's approximation

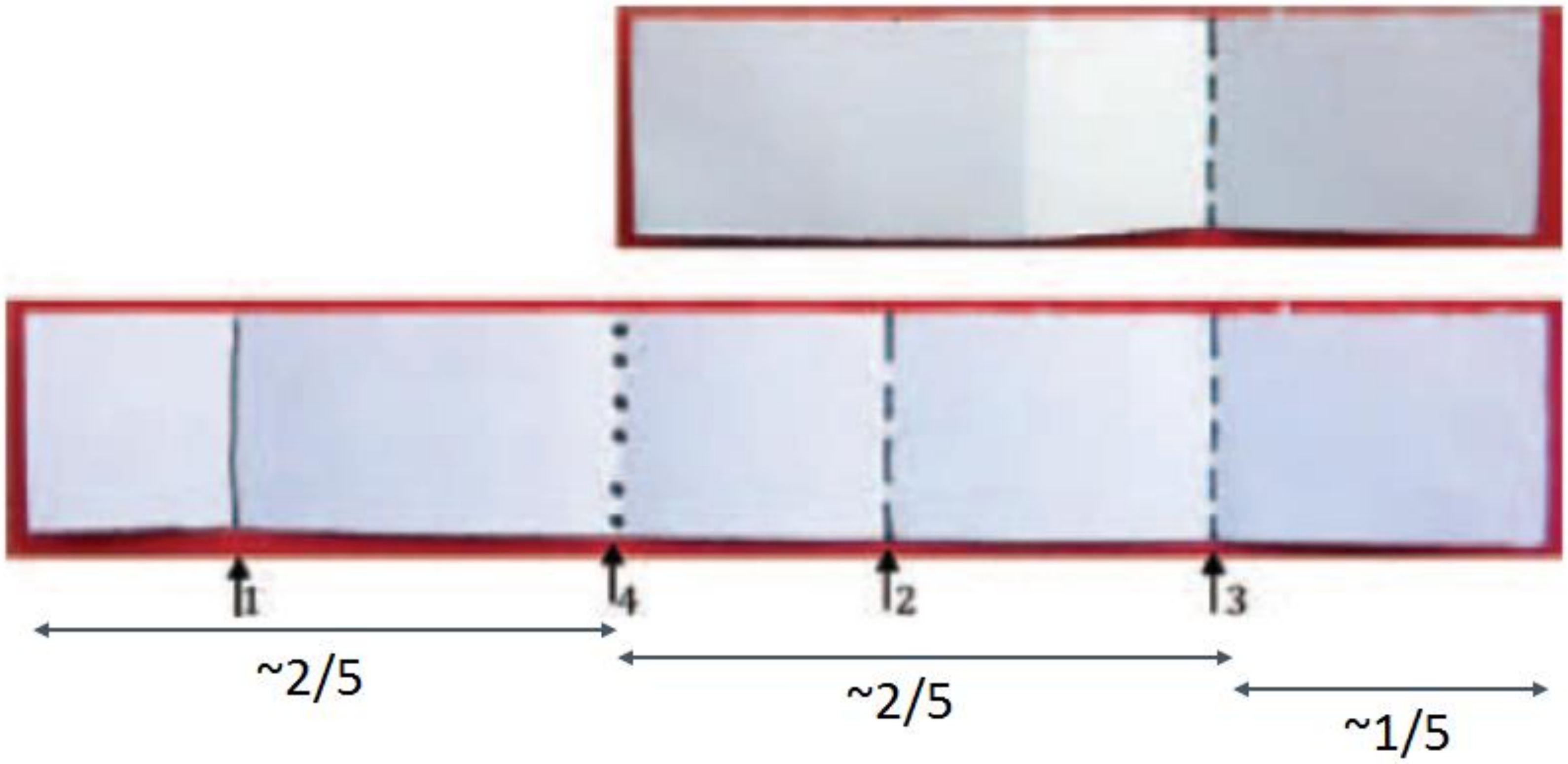
› Step 3: The pinch 2 is near 3/5th mark and the right side is roughly 2/5 of the paper.   Pinch the right side in half.



Source: http://www.teachersofindia.org/sites/default/files/5_how_do_you_divide_a_strip_into_equal_fifths.pdf

Fujimoto's approximation

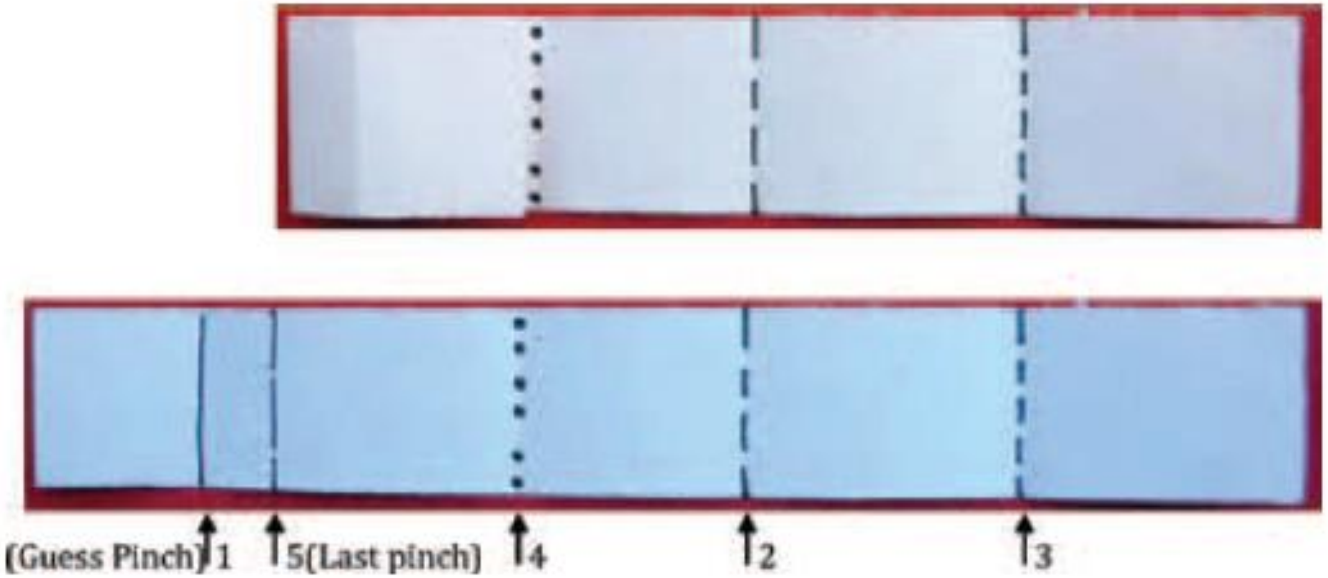
› Step 4: The pinch 3 is near 4/5th mark and the left side of is roughly 4/5 of the paper.   Pinch the left side in half.



Source: http://www.teachersofindia.org/sites/default/files/5_how_do_you_divide_a_strip_into_equal_fifths.pdf

Fujimoto’s approximation

- Step 5: The pinch 4 is near $2/5^{th}$ mark. Pinch the left side side in half. The last pinch is very close to the $1/5^{th}$ mark.

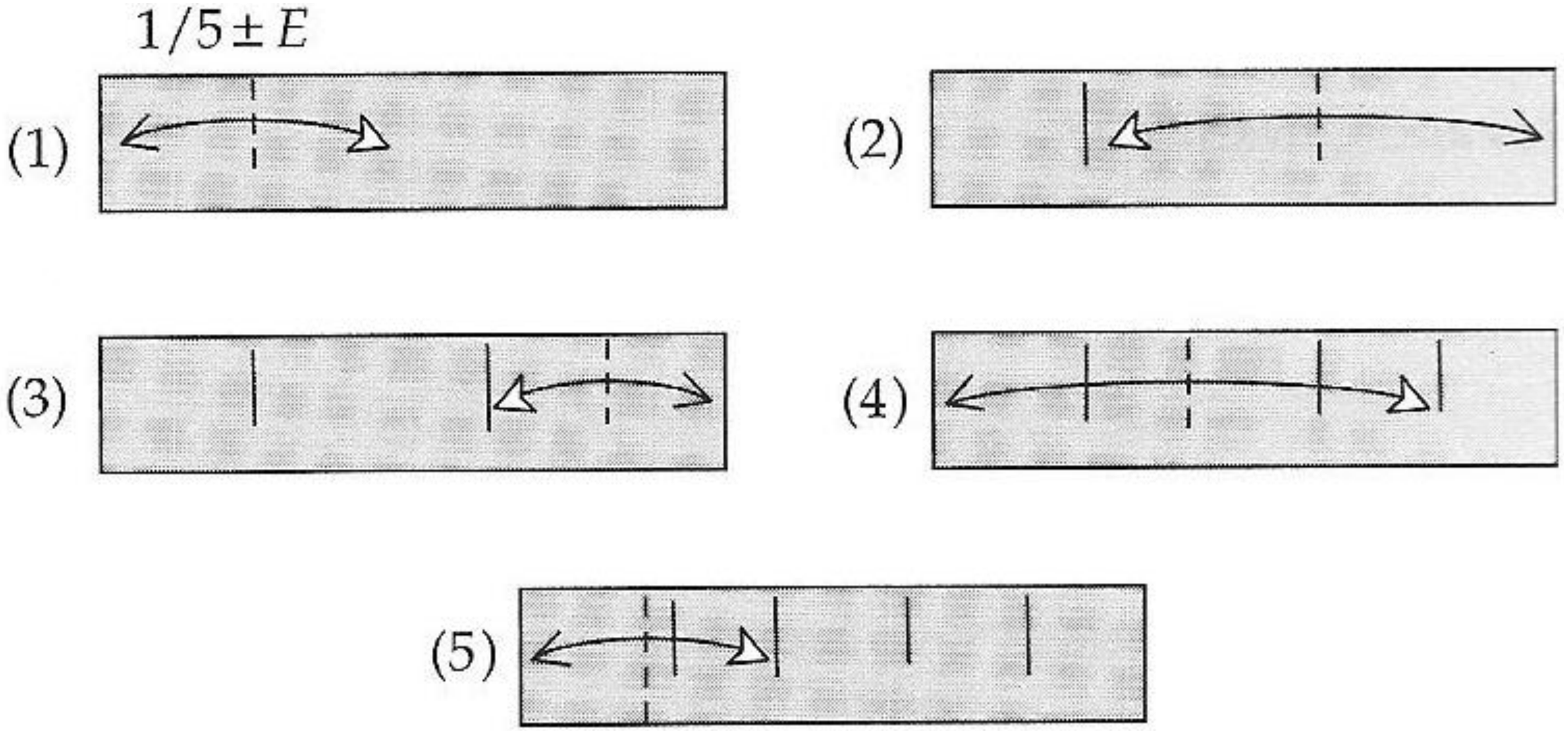


- Repeat steps to get better approximations

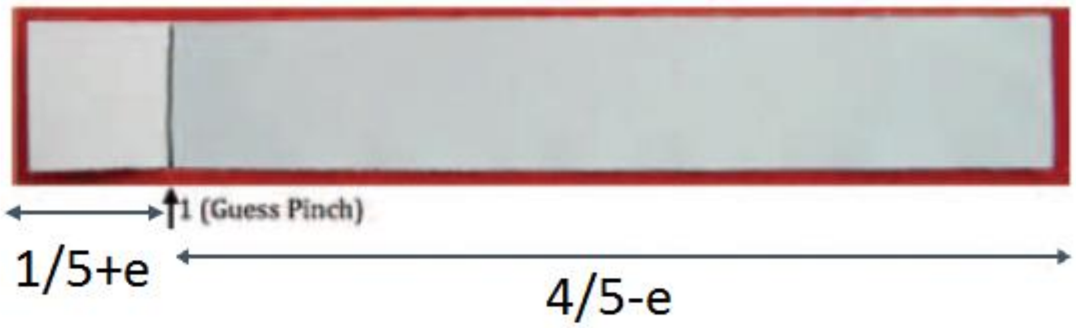
Source: http://www.teachersofindia.org/sites/default/files/5_how_do_you_divide_a_strip_into_equal_fifths.pdf

Fujimoto’s approximation

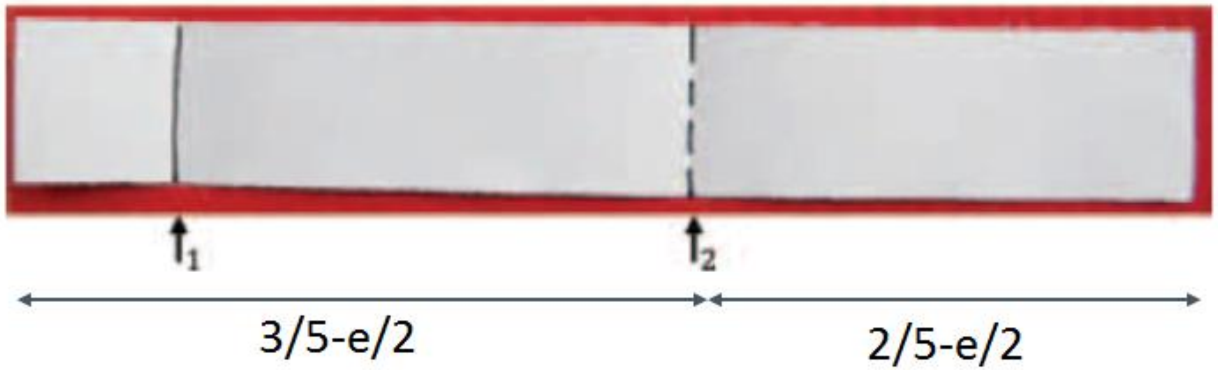
› Summary:



Why it works?

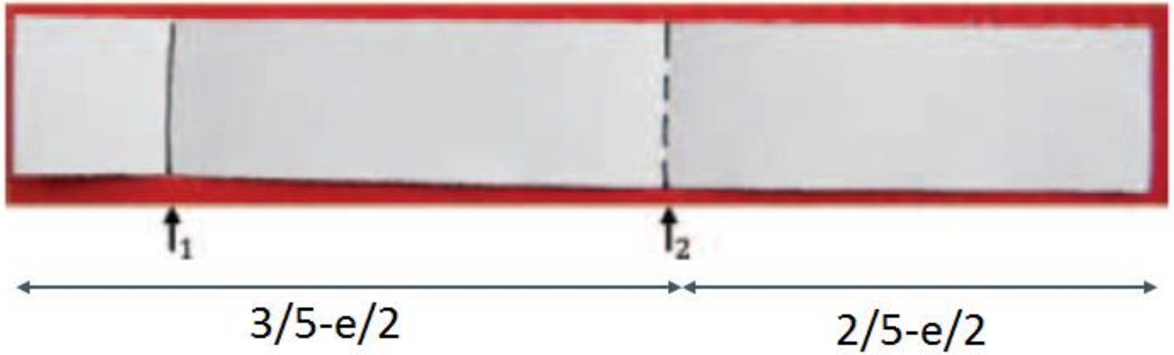


› First guess pinch is at $\frac{1}{5} + e$.
The right of pinch 1 has length $1 - \left(\frac{1}{5} + e\right) = \frac{4}{5} - e$.

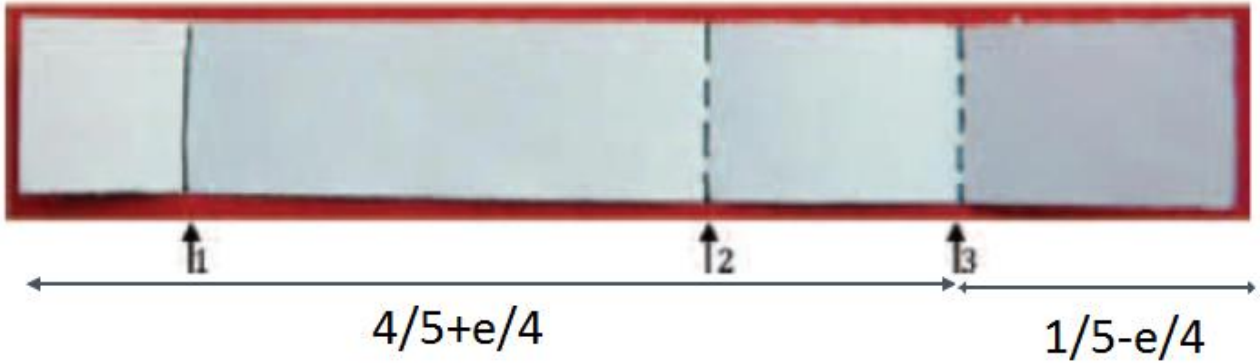


› Position of pinch 2 is $\left(\frac{1}{5} + e\right) + \frac{1}{2}\left(\frac{4}{5} - e\right) = \frac{3}{5} + \frac{e}{2}$.

Why it works?

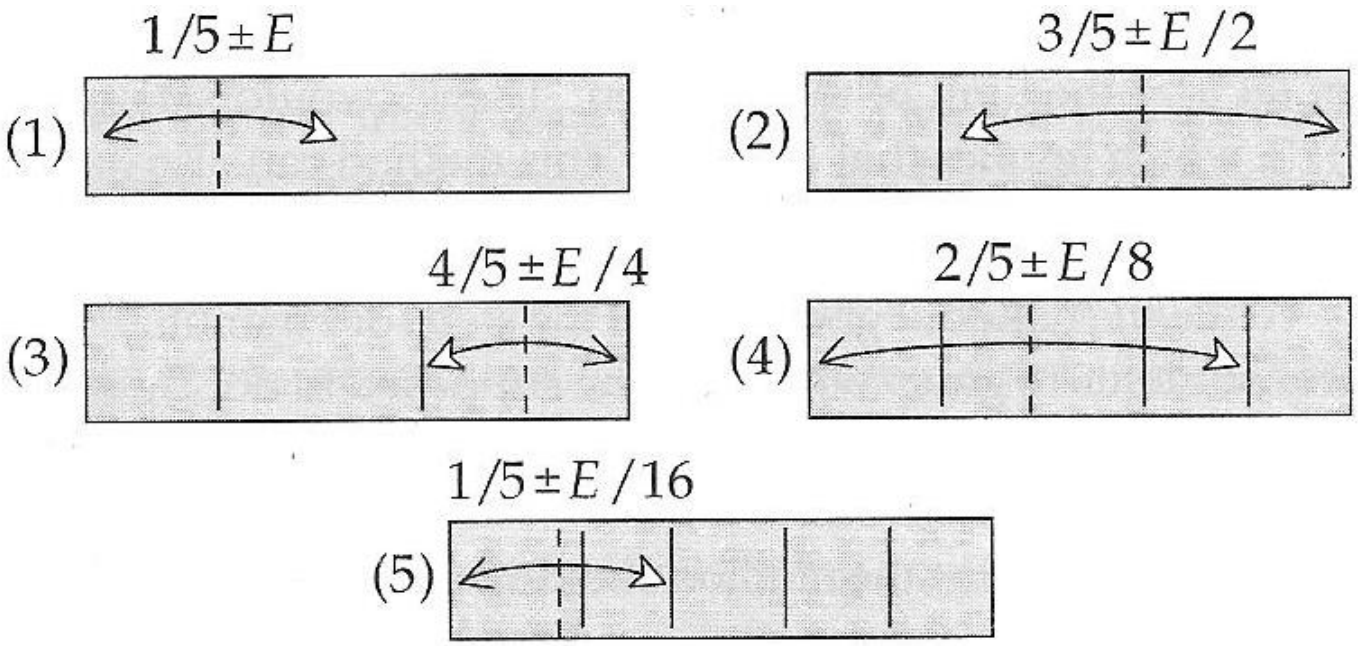


- Position of pinch 2 is $\left(\frac{1}{5}+e\right)+\frac{1}{2}\left(\frac{4}{5}-e\right)=\frac{3}{5}+\frac{e}{2}$.



- Position of pinch 3 is $\left(\frac{3}{5}+\frac{e}{2}\right)+\frac{1}{2}\left(1-\left(\frac{3}{5}+\frac{e}{2}\right)\right)=\frac{4}{5}+\frac{e}{4}$.
- With each fold, the error e is halved.

Why it works?



- More calculation shows at the fifth pinch, we are at the position $\frac{1}{5} + \frac{e}{16}$.
- Doing one round of Fujimoto approximation reduces the error by a factor of 16.

Some mathematics….

- A base 2 representation (其二進製) of a fraction (分數) $0 < x < 1$ is

$$x = \frac{i_1}{2} + \frac{i_2}{4} + \frac{i_3}{8} + \frac{i_4}{16} + \frac{i_5}{32} + \frac{i_6}{64} + \cdots = \sum_{j=1}^{\infty} \frac{i_j}{2^j} ,$$

where $i_j = 0$ or 1.

- $\frac{1}{5} < \frac{1}{2} \quad \rightarrow i_1 = 0,$
- $\frac{1}{5} < \frac{1}{4} \quad \rightarrow i_2 = 0,$
- $\frac{1}{5} = \frac{8}{40} > \frac{1}{8} = \frac{5}{40} \quad \rightarrow i_3 = 1,$

Some mathematics....

- The base 2 representation (其二進製) of $\frac{1}{5}$ is

$$\frac{1}{5} = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{i_4}{16} + \frac{i_5}{32} + \frac{i_6}{64} + \cdots$$

- $\frac{1}{5} = \frac{16}{80} > \frac{1}{8} + \frac{1}{16} = \frac{15}{80} \quad \rightarrow i_4 = 1,$
- $\frac{1}{5} = \frac{32}{160} < \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{35}{160} \quad \rightarrow i_5 = 0 \ldots$

Base 2 representation of $\frac{1}{5}$

- So, the base 2 representation (其二進製) of $\frac{1}{5}$ is

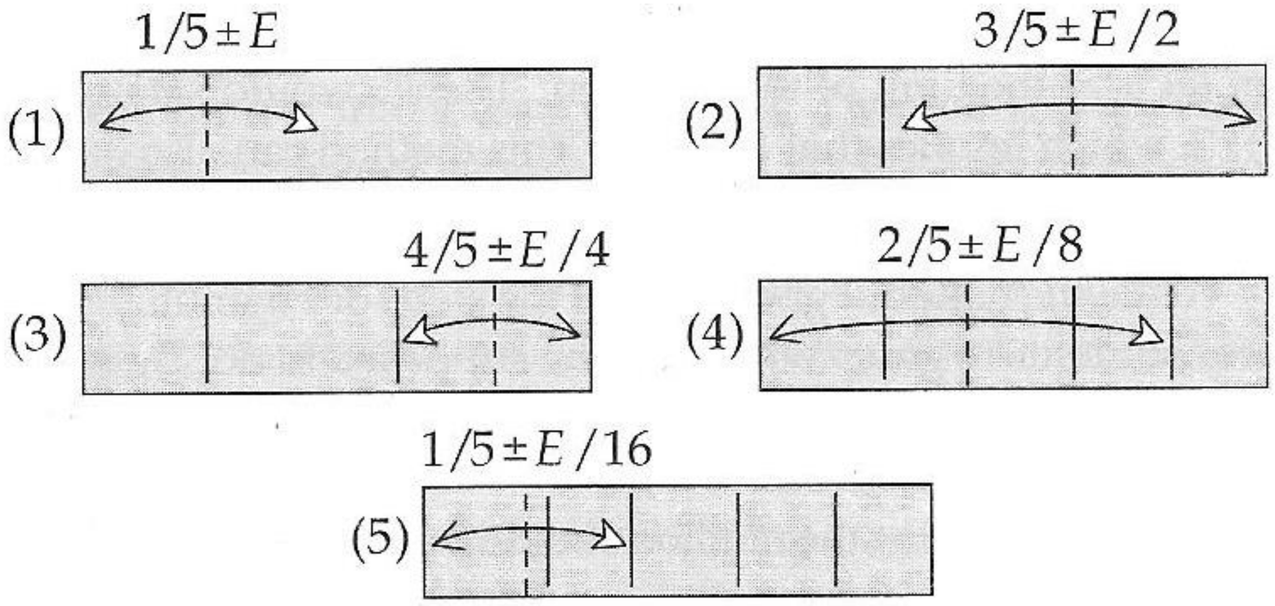
$$\frac{1}{5} = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{1}{16} + \frac{0}{32} + \frac{0}{64} + \cdots = (0.\overline{0011})_2$$

- Label 1 as folding **right** and 0 as folding **left**, then reading backwards, we do:

(right x 2)-(left x 2)-(right x 2)-(left x 2) …

Base 2 representation of $\frac{1}{5}$

- (right x 2)-(left x 2)-(right x 2)-(left x 2) ...
- Exactly as the Fujimoto's 1/5th approximation:



- COINCIDENCE?

Fujimoto's approximation for equal $\frac{1}{3}$th

› Another example

$$\frac{1}{3} = \frac{i_1}{2} + \frac{i_2}{4} + \frac{i_3}{8} + \frac{i_4}{16} + \frac{i_5}{32} + \frac{i_6}{64} + \cdots$$

› $\frac{1}{3} < \frac{1}{2} \quad \rightarrow i_1 = 0,$

› $\frac{1}{3} > \frac{1}{4} \quad \rightarrow i_2 = 1,$

› $\frac{1}{3} = \frac{8}{24} < \frac{1}{4} + \frac{1}{8} = \frac{9}{24} \quad \rightarrow i_3 = 0,$

Fujimoto's approximation for equal $\frac{1}{3}$th

- Another example

$$\frac{1}{3} = \frac{0}{2} + \frac{1}{4} + \frac{0}{8} + \frac{i_4}{16} + \frac{i_5}{32} + \frac{i_6}{64} + \cdots$$

- $\frac{1}{3} = \frac{8}{24} < \frac{1}{4} + \frac{1}{8} = \frac{9}{24} \quad \rightarrow i_3 = 0,$
- $\frac{1}{3} = \frac{16}{48} > \frac{1}{4} + \frac{1}{16} = \frac{15}{48} \quad \rightarrow i_4 = 1,$
- $\frac{1}{3} = \frac{32}{96} < \frac{1}{4} + \frac{1}{16} + \frac{1}{32} = \frac{33}{96} \quad \rightarrow i_5 = 0 \ldots$

Fujimoto's approximation for equal $\frac{1}{3}$th

› So, the base 2 representation of $\frac{1}{3}$ is

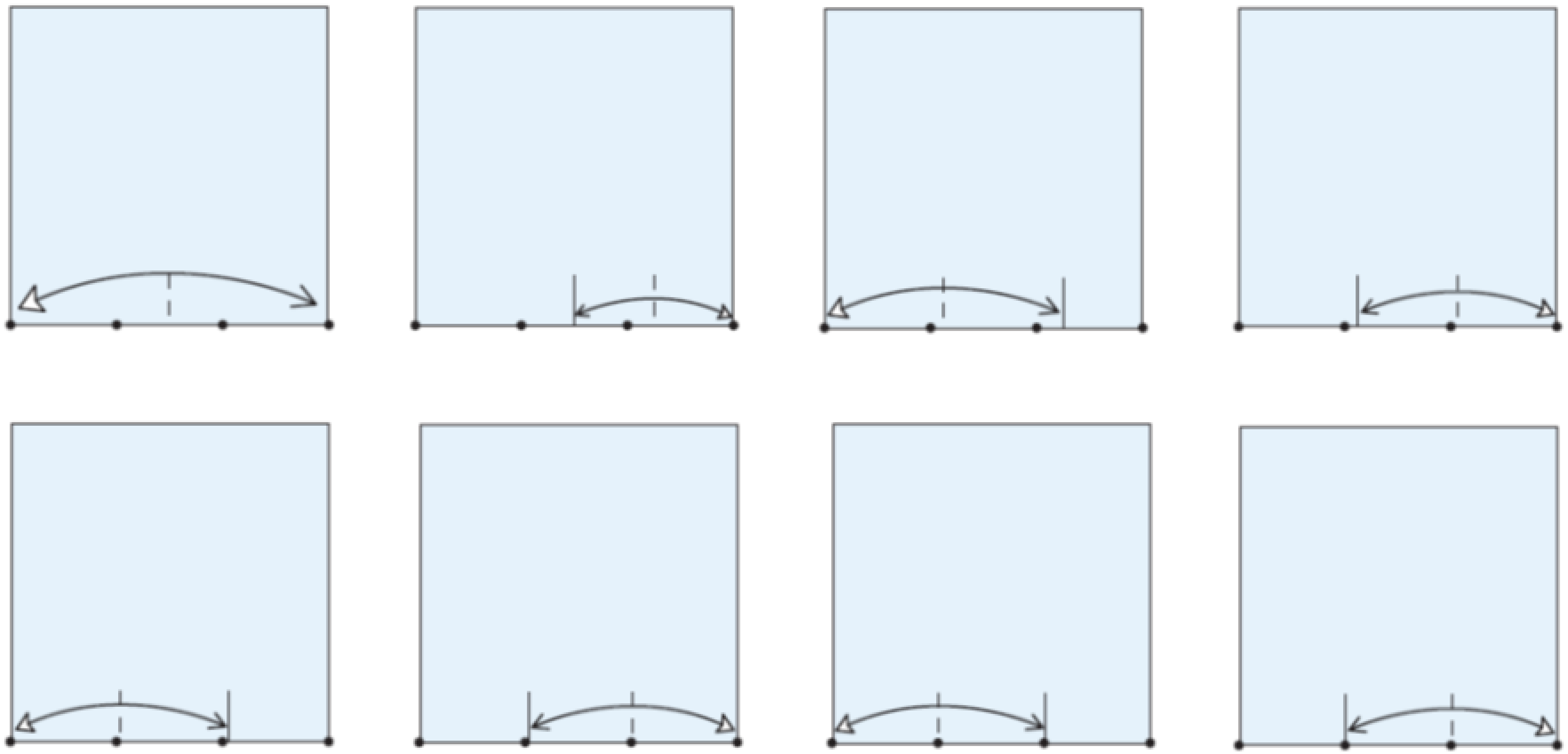
$$\frac{1}{3} = \frac{0}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} + \frac{0}{32} + \frac{1}{64} + \cdots = (0.\overline{01})_2$$

› Label 1 as folding right and 0 as folding left, then reading backwards, the action is:

(right)-(left)-(right)-(left) ...

Example: $\frac{1}{3} = (0.\overline{01})_2$

› After initial guess, we fold right-left-right-left-right-left-….



Source: Robert Lang – Origami and Construction

Another example

$$\frac{1}{7} = \frac{i_1}{2} + \frac{i_2}{4} + \frac{i_3}{8} + \frac{i_4}{16} + \frac{i_5}{32} + \frac{i_6}{64} + \cdots$$

- $\frac{1}{7} < \frac{1}{2} \quad \rightarrow i_1 = 0,$
- $\frac{1}{7} < \frac{1}{4} \quad \rightarrow i_2 = 0,$
- $\frac{1}{7} > \frac{1}{8} \quad \rightarrow i_3 = 1,$

Another example

$$\frac{1}{7} = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{i_4}{16} + \frac{i_5}{32} + \frac{i_6}{64} + \cdots$$

- $\frac{1}{7} = \frac{16}{112} < \frac{1}{8} + \frac{1}{16} = \frac{21}{112} \qquad \rightarrow i_4 = 0,$
- $\frac{1}{7} = \frac{32}{224} < \frac{1}{8} + \frac{1}{32} = \frac{35}{224} \qquad \rightarrow i_5 = 0,$
- $\frac{1}{7} = \frac{64}{448} > \frac{1}{8} + \frac{1}{64} = \frac{63}{448} \qquad \rightarrow i_6 = 1,$

Fujimoto's approximation for equal $\frac{1}{7}$th

- So, the base 2 representation of $\frac{1}{7}$ is

$$\frac{1}{7} = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{0}{16} + \frac{0}{32} + \frac{1}{64} + \cdots = (0.\overline{001})_2$$

- Label 1 as folding **right** and 0 as folding **left**, then reading backwards, the action is:

(right)-(left x2)-(right)-(left x2) …

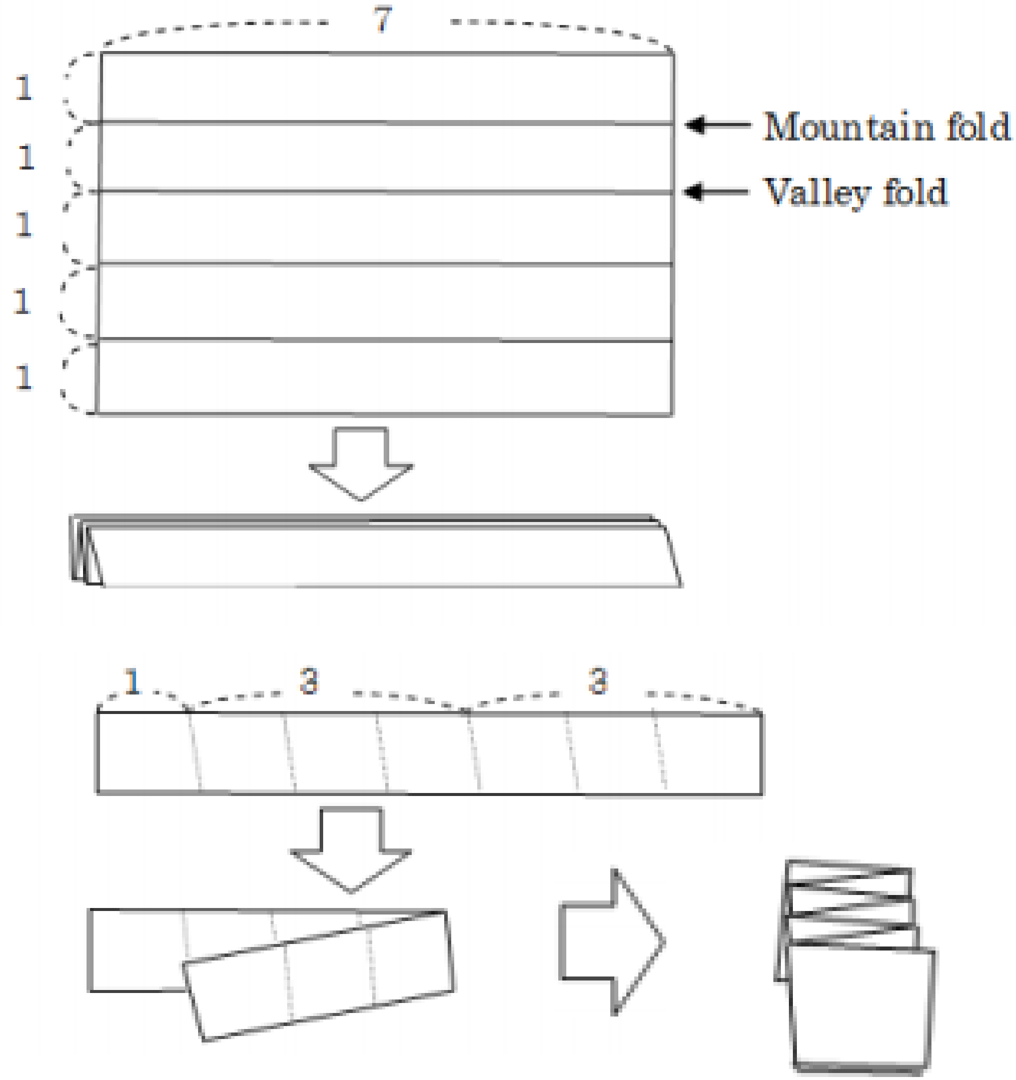
General (一般性) procedure for $\frac{1}{N}$

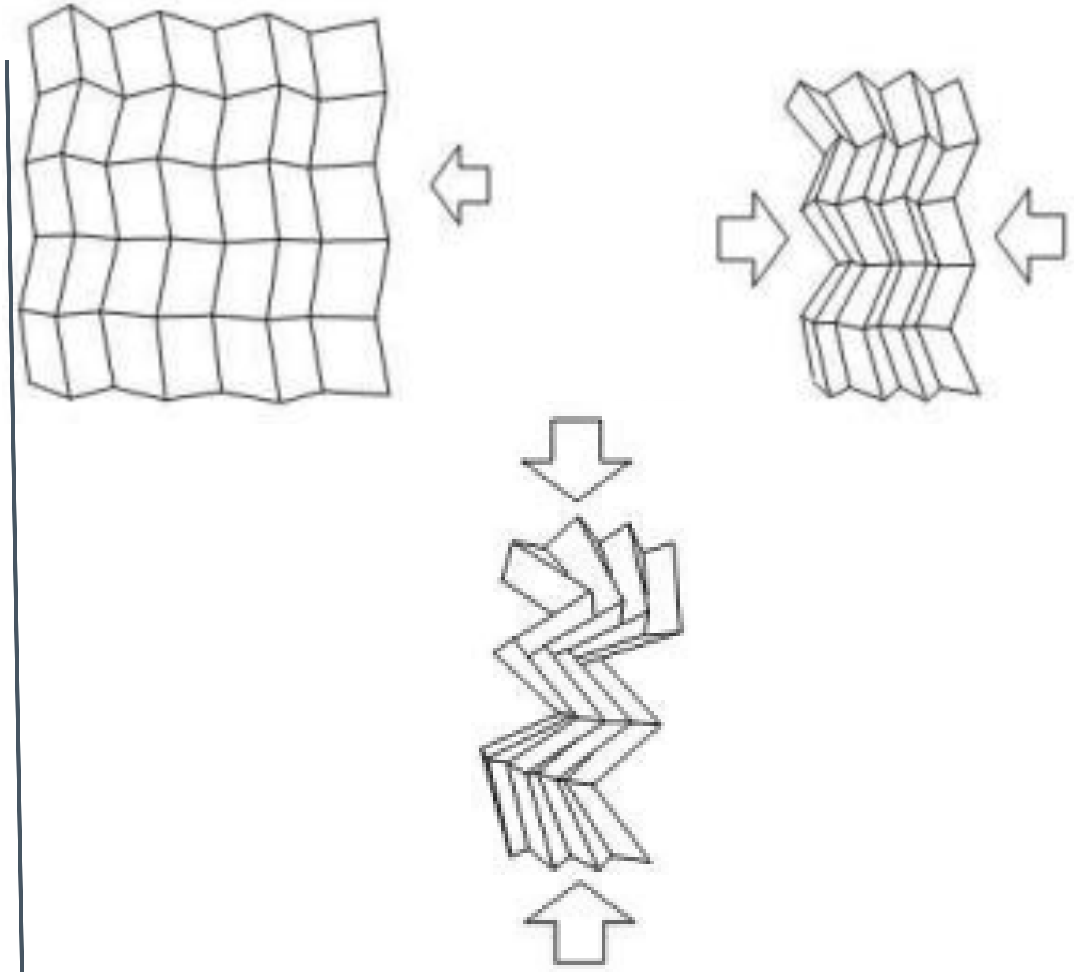
- **Step 1**: Write $\frac{1}{N}$ in terms of the binary representation.
- Example:

$$\frac{1}{3} = (0.\overline{01})_2, \qquad \frac{1}{7} = (0.\overline{001})_2, \qquad \frac{1}{9} = (0.\overline{000111})_2$$

- **Step 2**: Make guess at where the $\frac{1}{N}$th mark is.
- **Step 3**: Following the binary representation after the decimal point reading backwards. Fold left for 0 and fold right for 1.

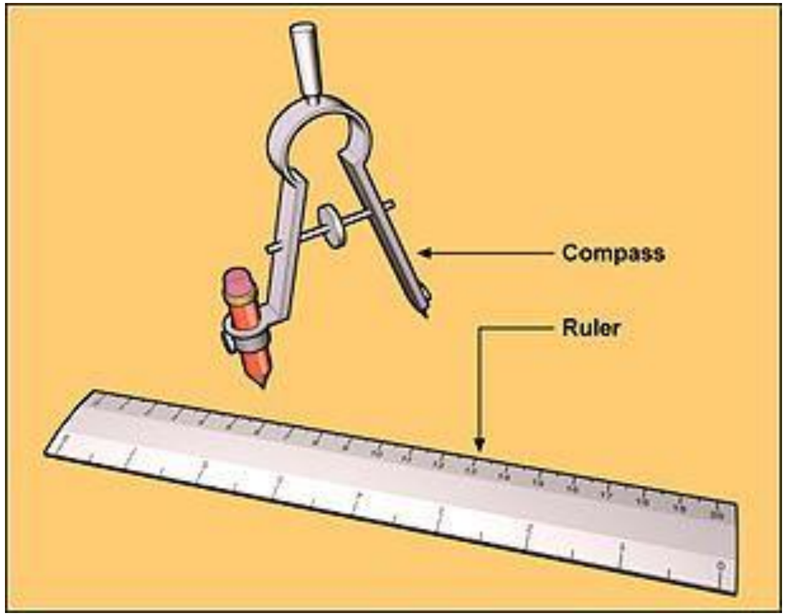
Application to Miura map folding

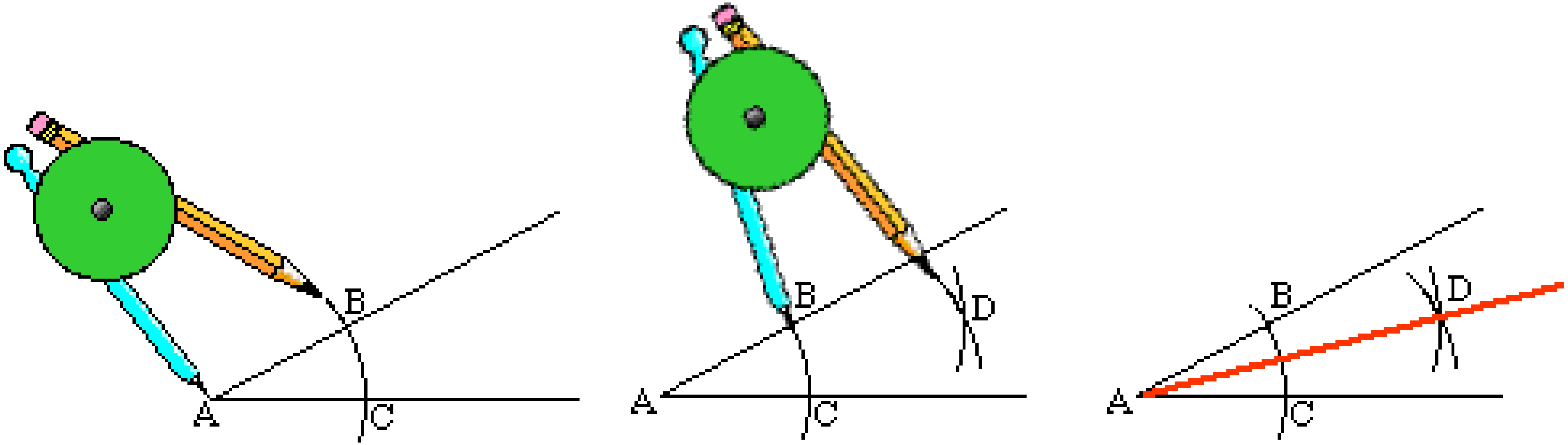




Angle bisection (角度平分)

- Problem: Use only compass (圓規) and a straight edge (直尺) to divide an angle θ into half.





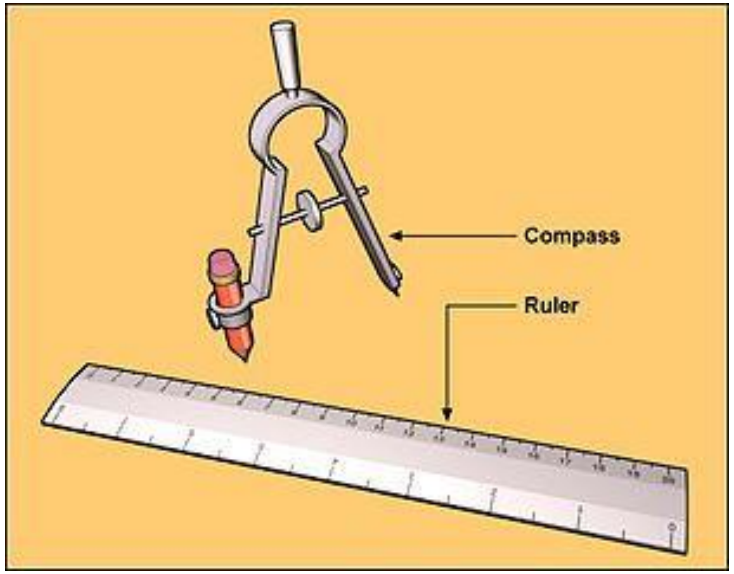
Source:
http://mathforum.org/sanders/geometry/GP05Constructions.html

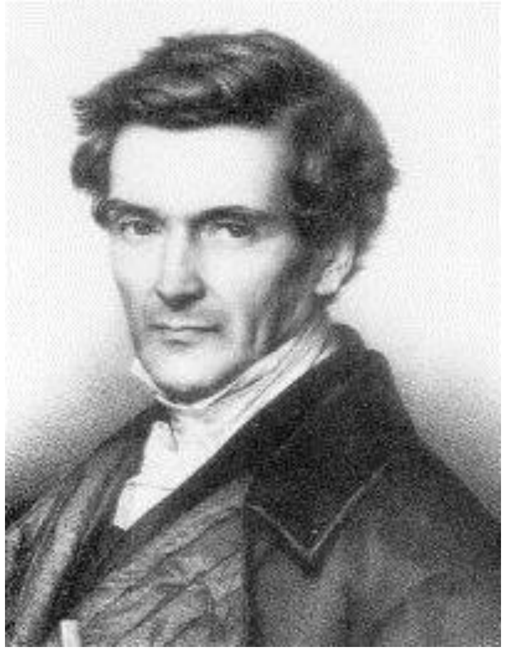
Angle trisection (角度三平分)

› Problem: Use only compass and a straight edge to divide an angle θ into equal thirds.

› Pierre WANTZEL (1814-1848) showed that it is <u>**impossible**</u> to do it with only a compass and a straight edge.

› But we can do this with origami!



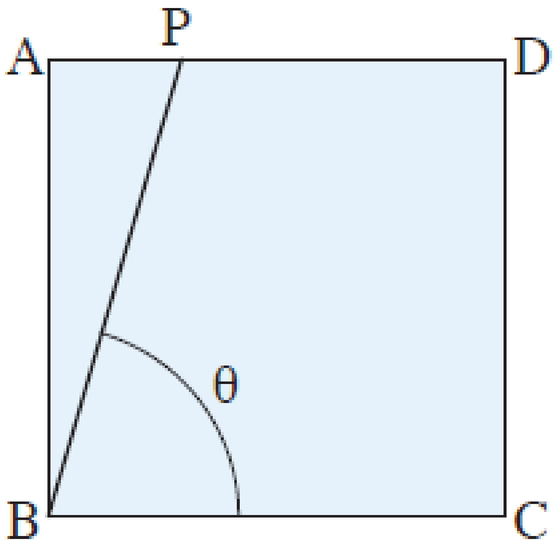


Source: https://3010tangents.wordpress.com/tag/pierre-wantzel/
https://paginas.matem.unam.mx/cprieto/biografias-de-matematicos-u-z/237-wantzel-pierre-laurent

Acute angle (鋭角) trisection with folding

› ABE Tsune developed a method for angles $\theta < 90°$.

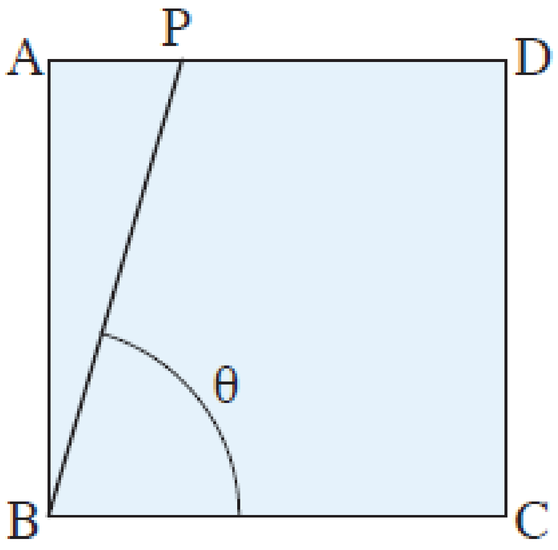
› Setting: Trisect the angle PBC

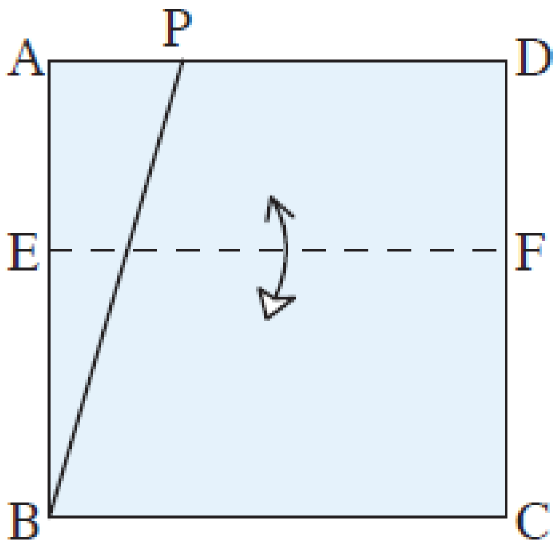


Source: Robert Lang – Origami and Construction

(Acute) Angle trisection with folding

› Step 1: Fold any line parallel (平行) to *BC* and create newline *EF*

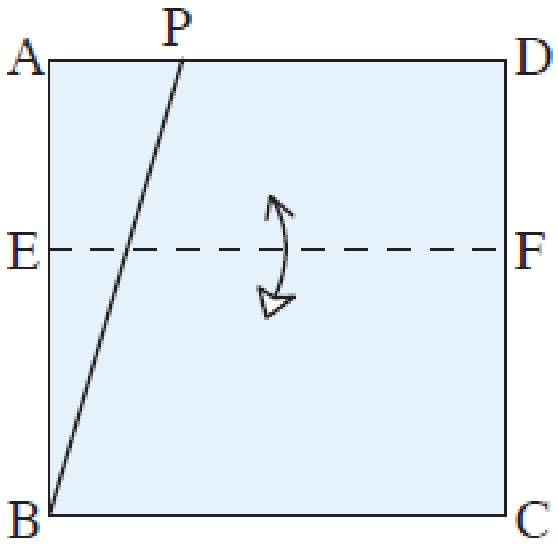


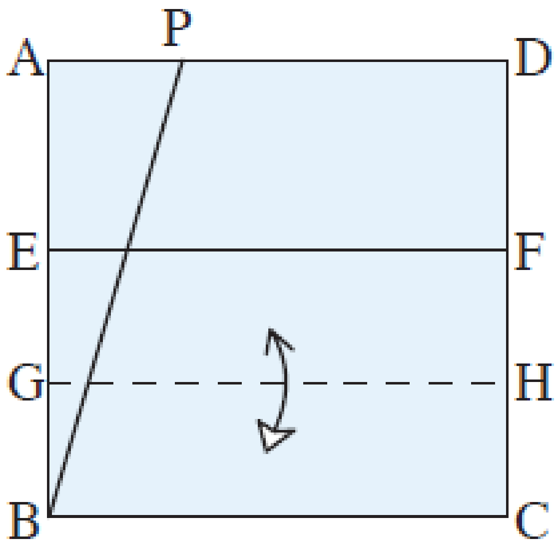


Source: Robert Lang – Origami and Construction

(Acute) Angle trisection with folding

- Step 2: Fold *BC* to *EF* to create newline *GH*. Then *BG*, *GE*, *CH* and *HF* have the same length.

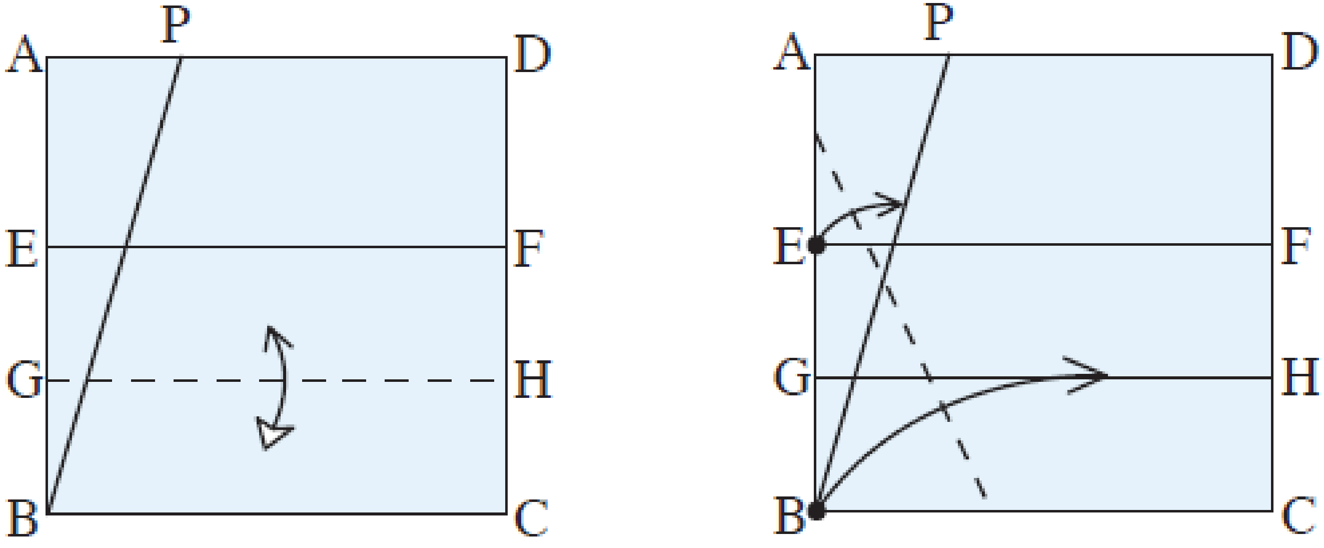




Source: Robert Lang – Origami and Construction

(Acute) Angle trisection with folding

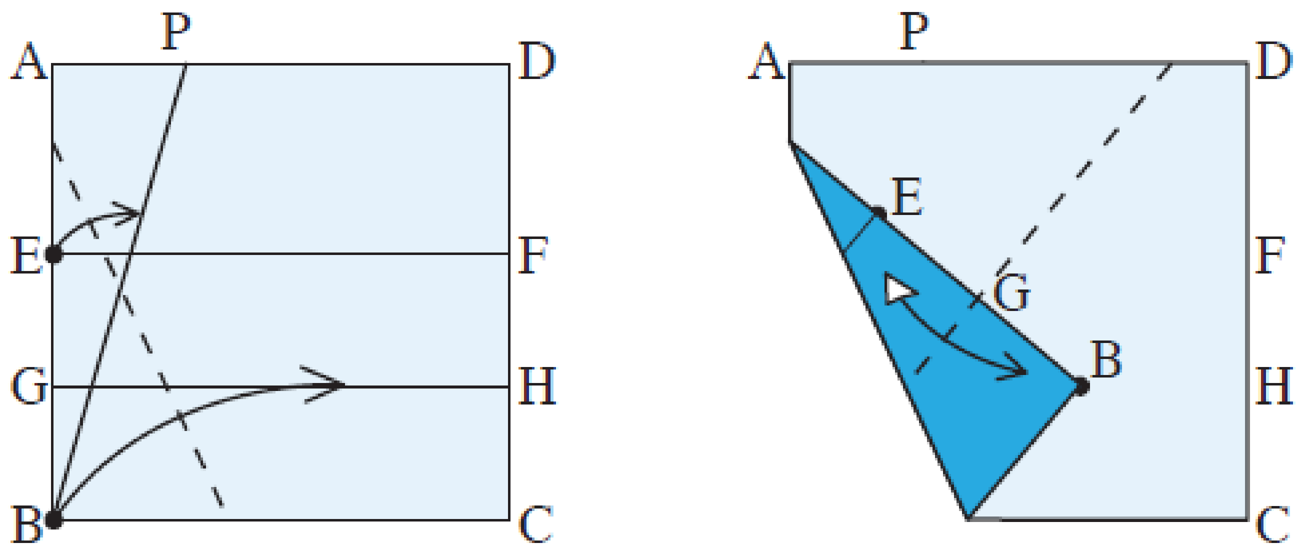
› Step 3: Fold corner *B* so that point *E* is on line *BP* and point *B* is on the line *GH*.



Source: Robert Lang – Origami and Construction

(Acute) Angle trisection with folding

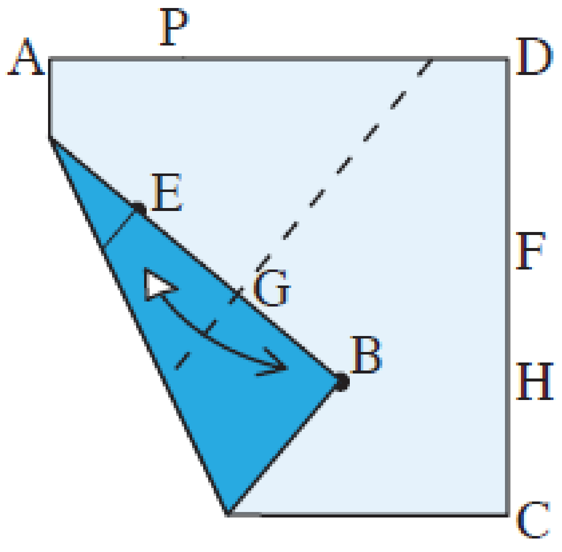
› Step 4: Create a new line by folding B to E. Then unfold.

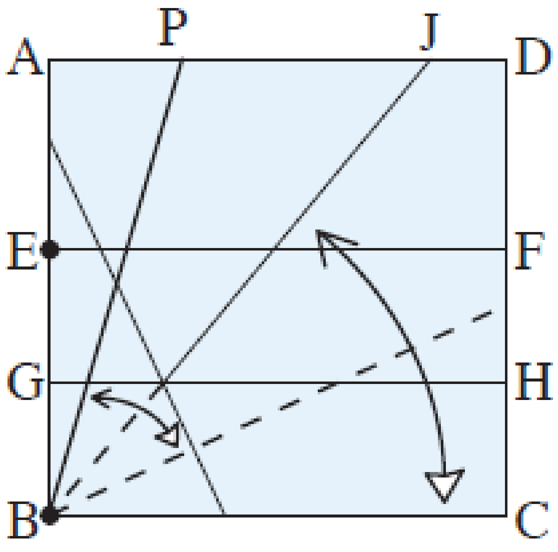


Source: Robert Lang – Origami and Construction

(Acute) Angle trisection with folding

- Step 5: Extend new line to get *BJ*, then bring line *BC* to *BJ*, and unfold again.

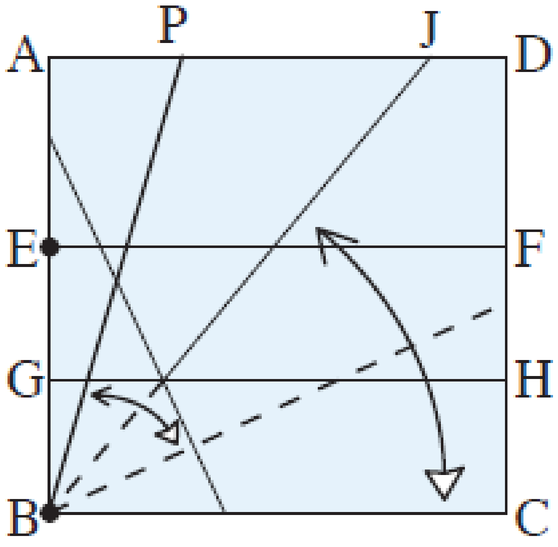


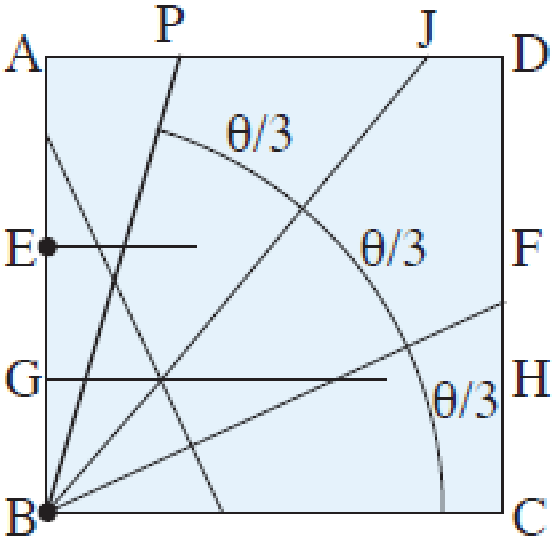


Source: Robert Lang – Origami and Construction

(Acute) Angle trisection with folding

› The angle is now trisected.

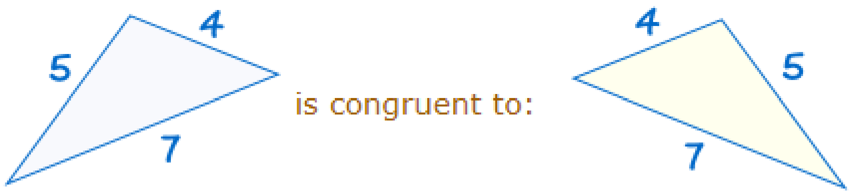


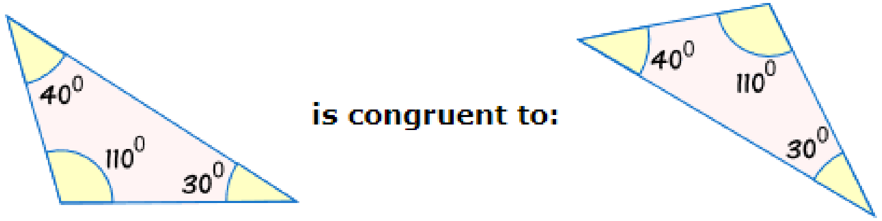


Source: Robert Lang – Origami and Construction

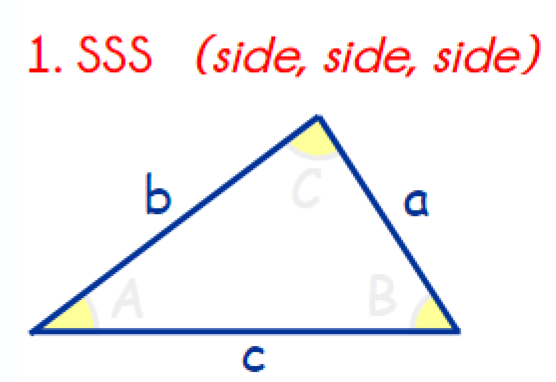
Revision – Congruent (全等) triangles

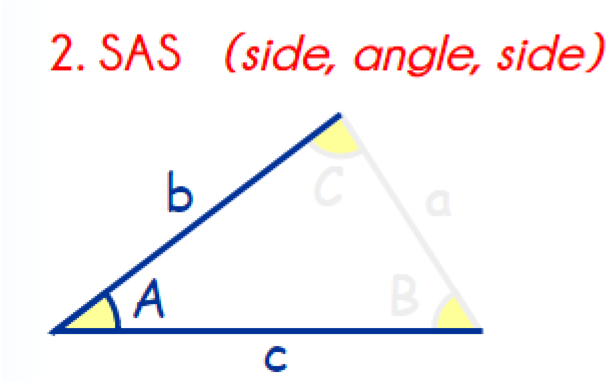
- Two triangles are **congruent** if they have the same three sides **and** exactly the same angles.





- Two criteria to check for congruence:

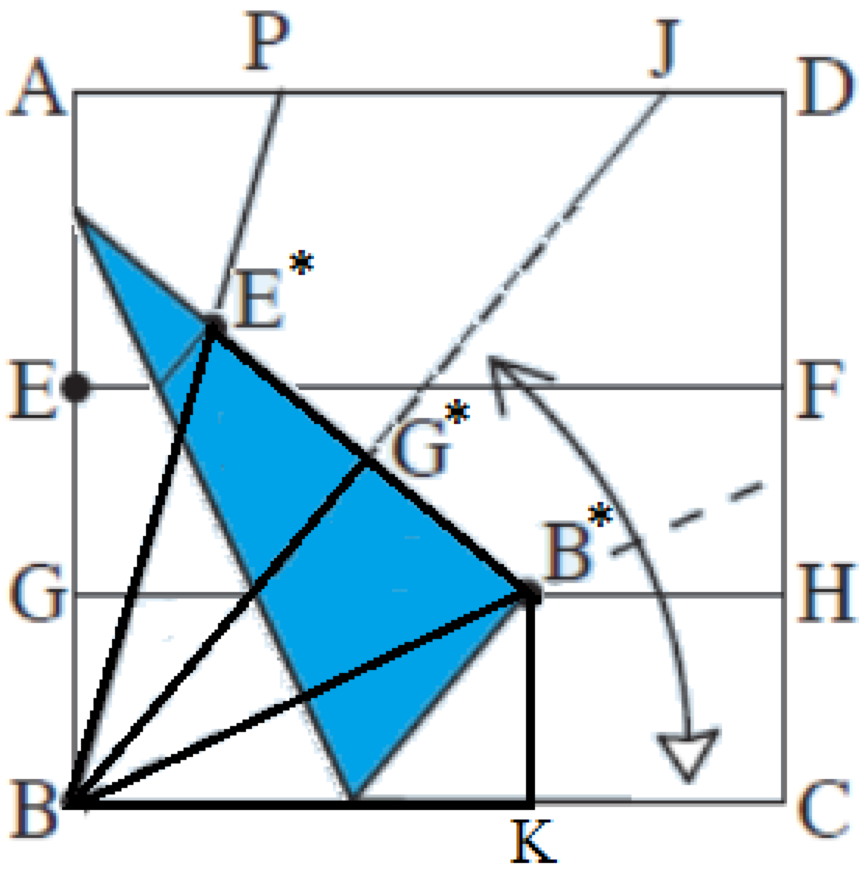




Source: https://www.mathsisfun.com/geometry/triangles-congruent.html

Why it works?

- 1) $EG = GB$ and so $E^*G^* = G^*B^*$.
- 2) BG^* perpendicular (垂直) to E^*B^*, so ΔBE^*G^* is congruent to ΔBG^*B^* (Side-Angle-Side).
- 3) $B^*K = G^*B^*$ and $BG^* = BK$, so ΔBG^*B^* is congruent to ΔBB^*K (Side-Side-Side)



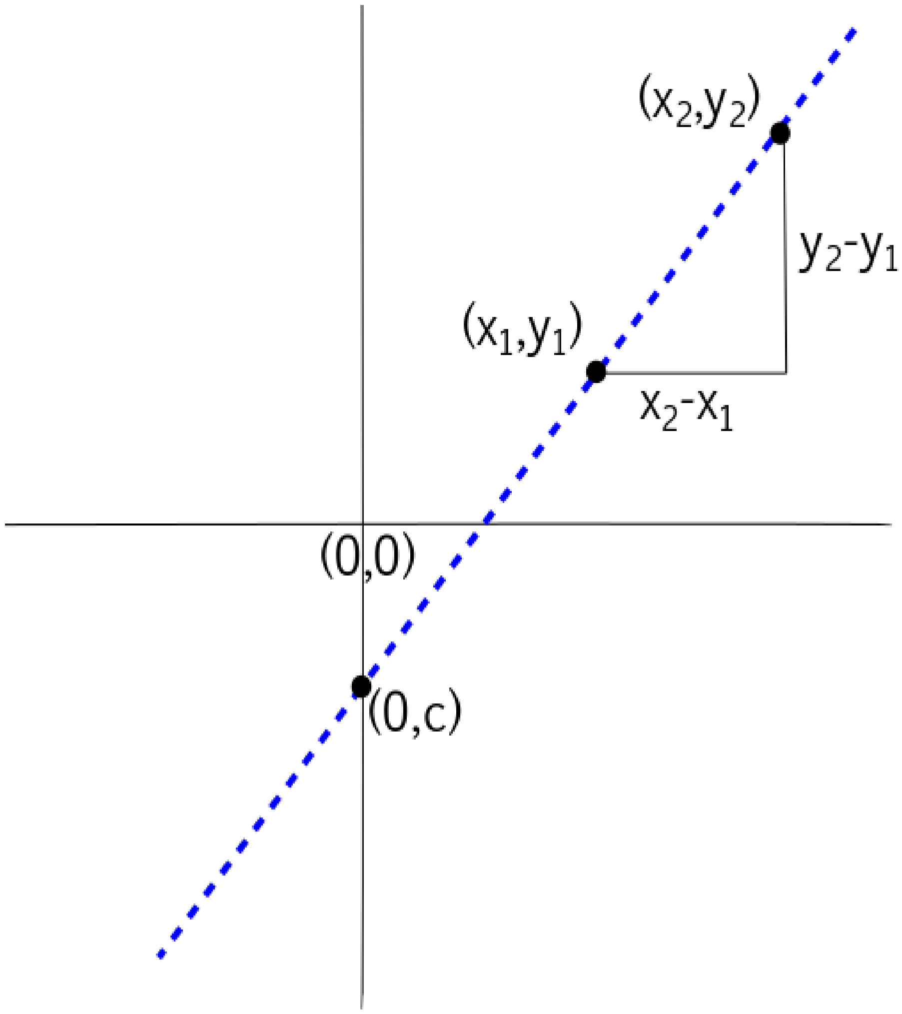
Revision – equation of a straight line

- Equation of a line is $y = mx + c$:
- m is the slope (坡度), and c is the intercept (交叉點).
- Two points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- And intercept is

$$c = y_1 - mx_1 = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1$$



Revision - quadratic equations (二次方程)

- Find solutions x to the equation

$$ax^2 + bx + c = 0$$

- This has only two solutions/roots (根) x_1 and x_2.
- Quadratic formula (二次公式)

$$x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

- Example: $x^2 - 1 = 0$ is $a = 1, b = 0, c = -1$ and has solutions $x_1 = 1$ and $x_2 = -1$.

Solving cubic equations (三次方程)

› Find solutions x to the equation

$$x^3 + ax^2 + bx + c = 0$$

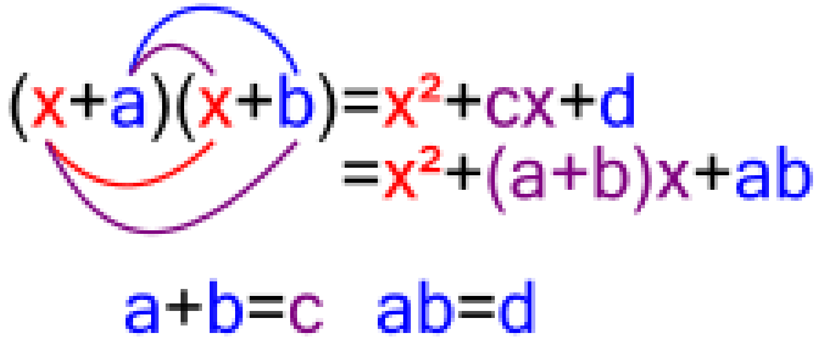
› This has three solutions/roots x_1, x_2 and x_3.

› Example: $x^3 + 3x^2 + 3x + 1 = 0$ has solutions $x_1 = x_2 = x_3 = -1$.

› Cubic formula (三次公式)?

(too complex)

Strategy



- Aim: To find **one root** x_1 to $x^3+ax^2+bx+c=0.$

- Then, factorize (因式分解)

$$x^3+ax^2+bx+c=(x-x_1)(x^2+ex+f)$$

- Use quadratic formula on $x^2+ex+f=0$ to find the other two roots x_2 and x_3.

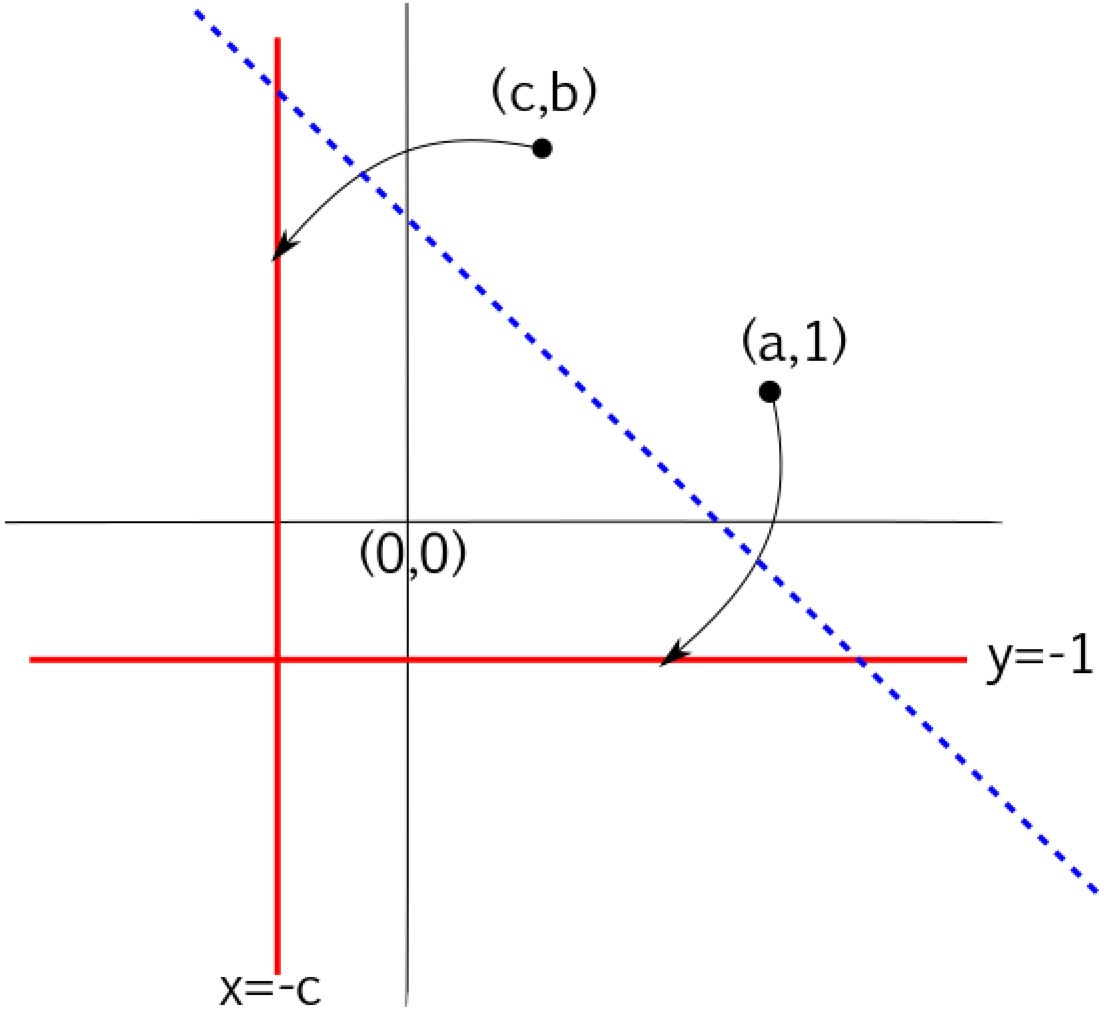
Source: https://en.wikipedia.org/wiki/Factorization

Solving $x^3+ax^2+bx+c=0$ with one fold

- Step 1.

  Mark the point $p_1 = (a,1)$ and $p_2 = (c,b)$ in the xy plane.

  Draw the lines $L_1 = \{y = -1\}$ and $L_2 = \{x = -c\}$.

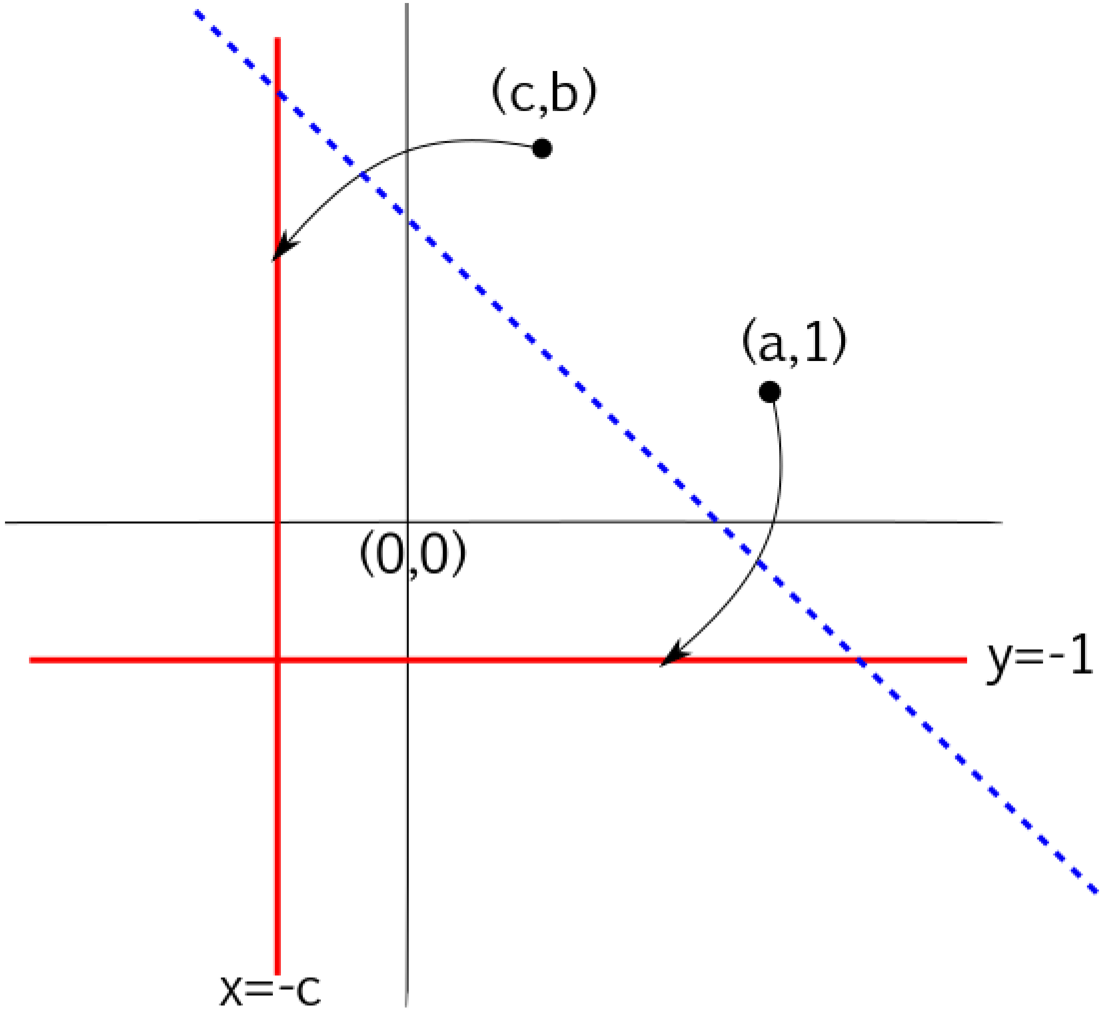


Solving $x^3+ax^2+bx+c=0$ with one fold

- Step 2.

  Fold p_1 to line L_1 and p_2 to line L_2 and create a new crease line.

- The **slope** of the crease line is a **root** to $x^3+ax^2+bx+c=0$.



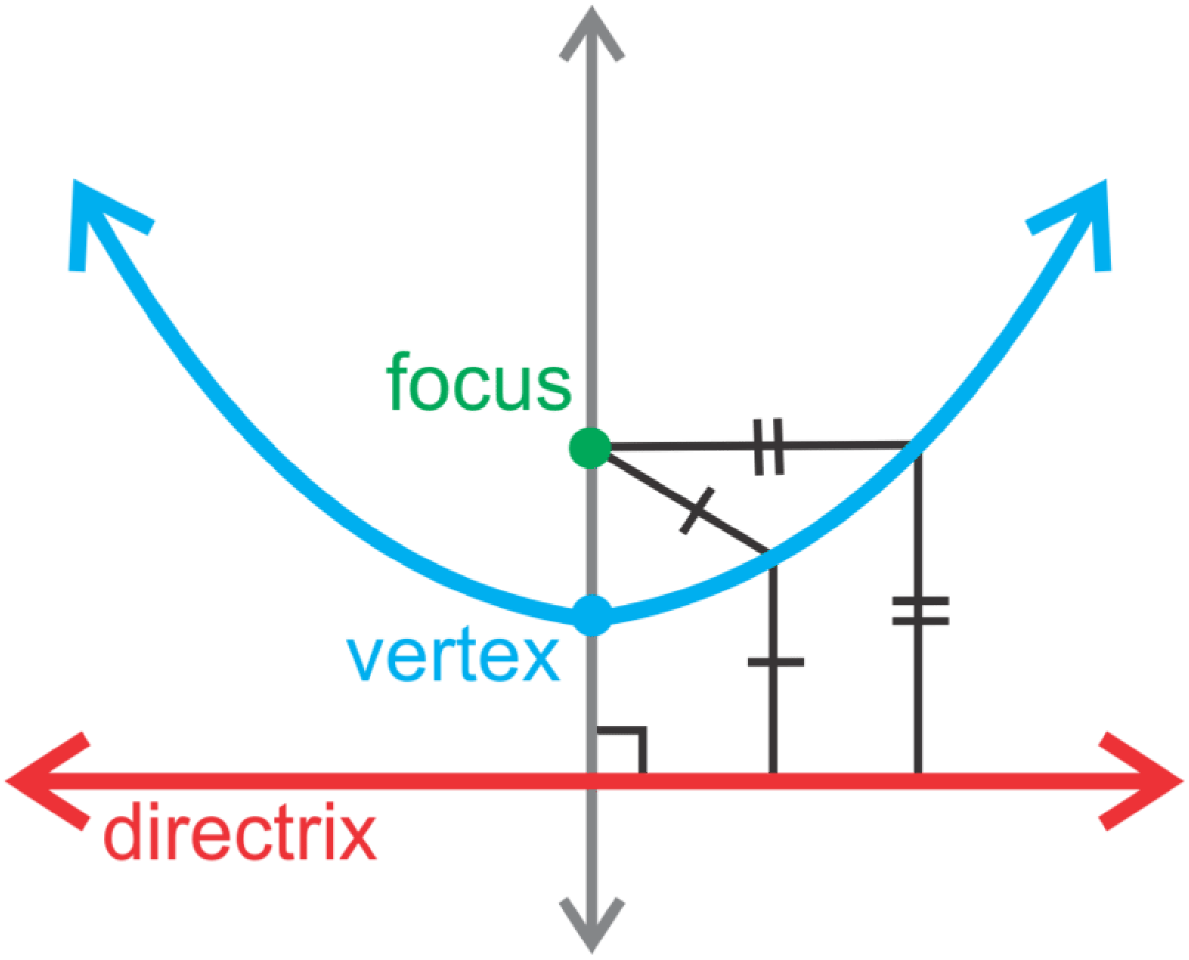
Explanation of why it works???

Maybe we just skip this…..

Folding a parabola

- Fix a point (the focus) and a line (the directrix).

- Draw a curve where the distance between the focus and the curve is the same as the distance between the directrix and the curve.



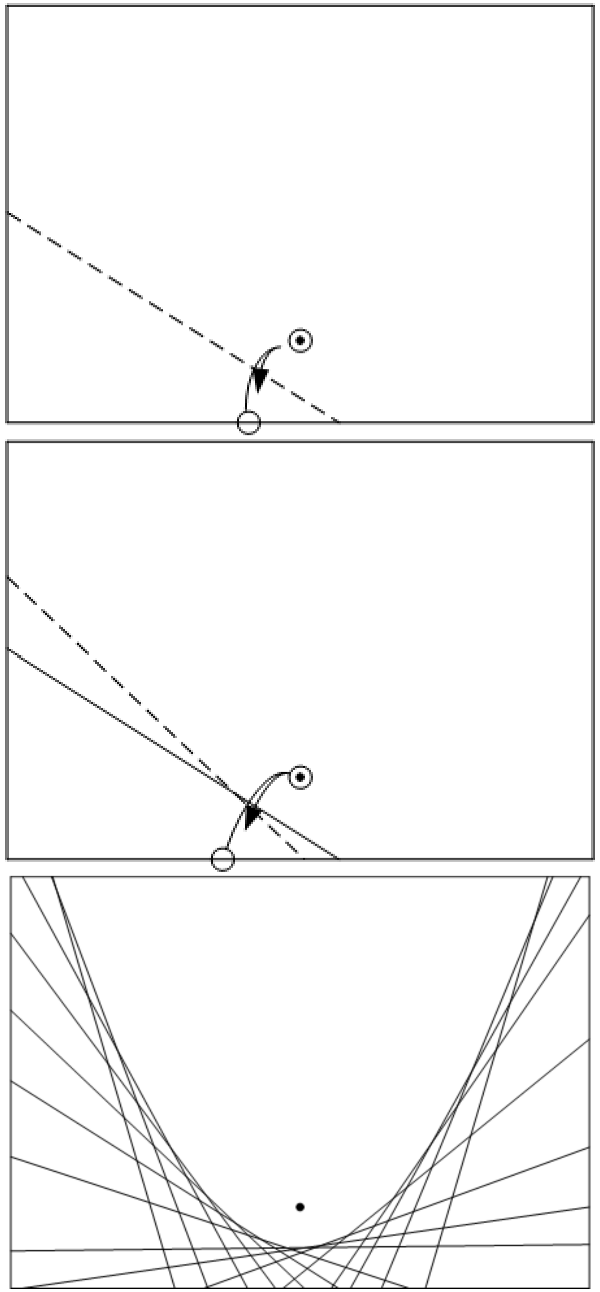
Source: www.ck12.org/calculus/parabolas-with-vertex-at-the-origin/lesson/Parabolas-with-Vertex-at-the-Origin-ALG-II/

Folding a parabola

› Mark one side of the paper as the directrix.

› Choose a point p anywhere inside the paper and fold the directrix to p over and over again.

› The point p will become the focus.



Source: http://www.cutoutfoldup.com/507-fold-a-parabola.php

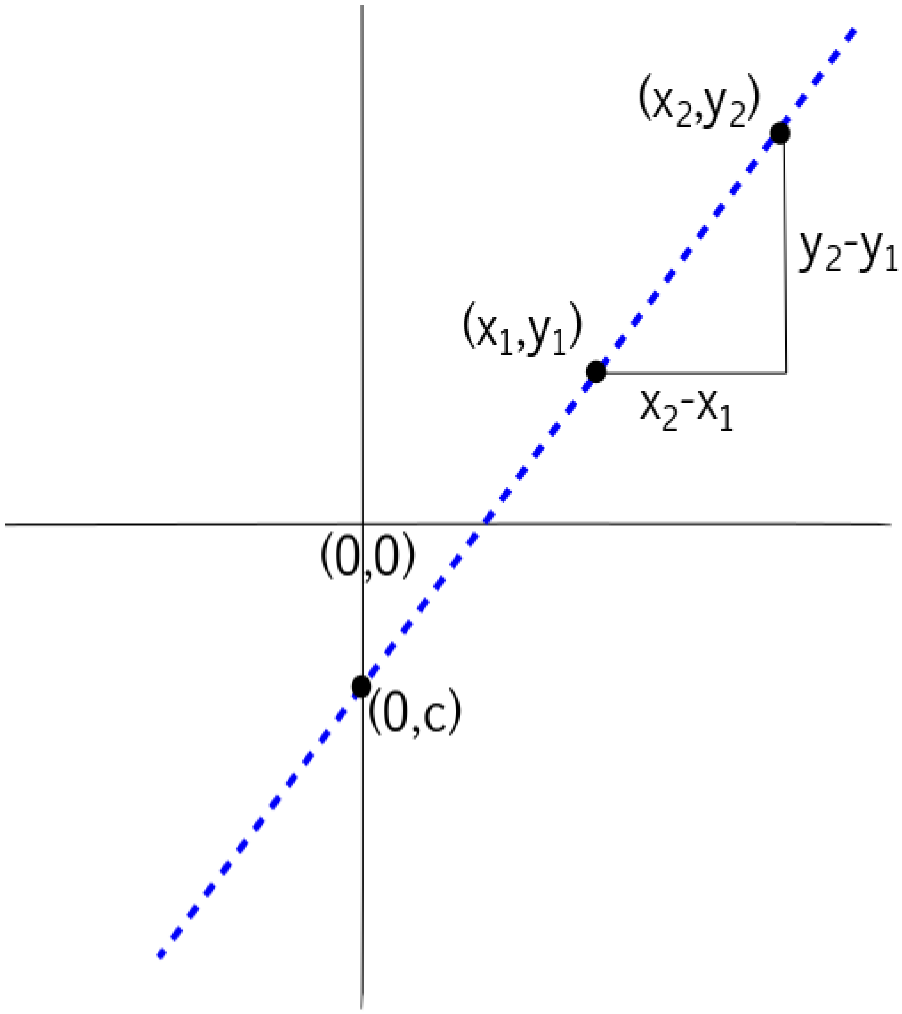
Revision – equation of a straight line

- Equation of a line is $y = mx + c$:
- m is the slope, and c is the intercept.
- Given two points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- And intercept is

$$c = y_1 - mx_1 = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1$$



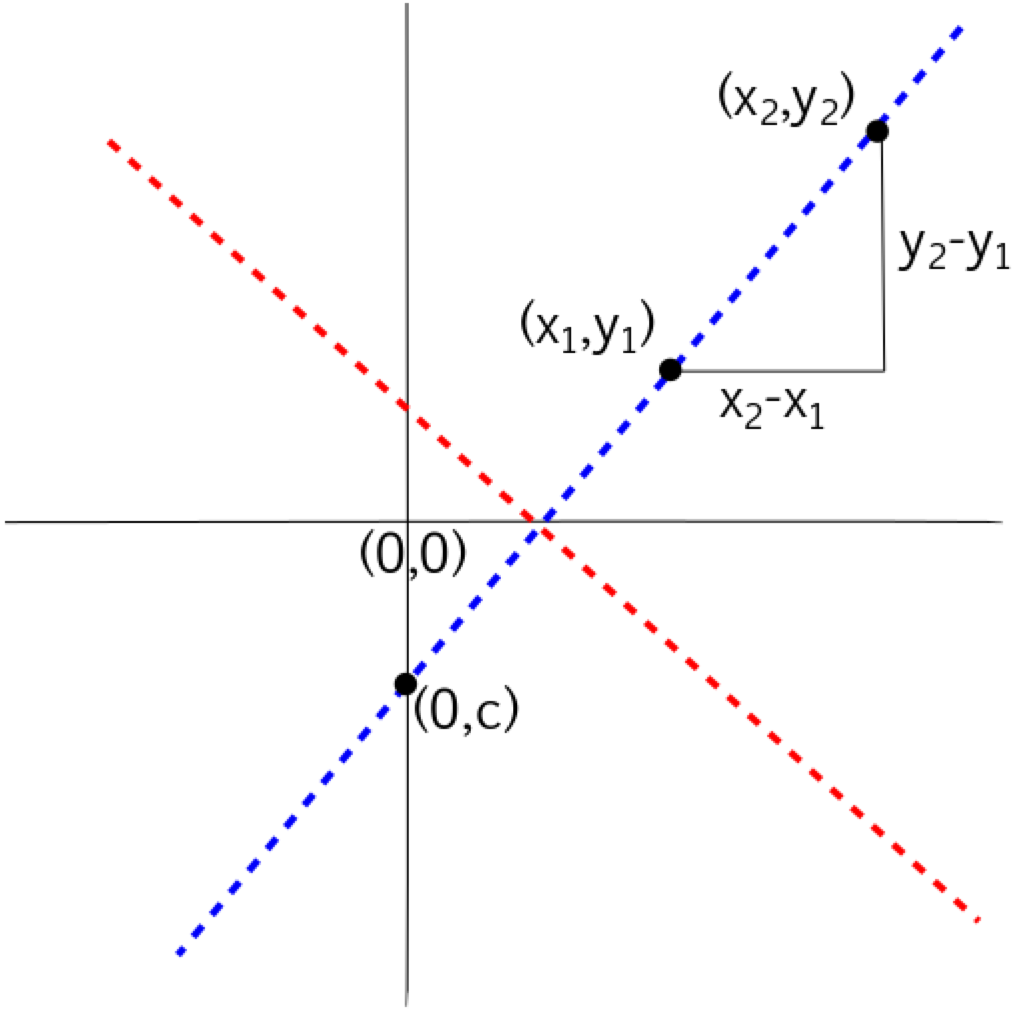
Revision – perpendicular lines

- Blue line has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Red line is perpendicular if the angles at the intersection are 90°
- So red line has slope

$$\frac{-1}{m} = \frac{x_1 - x_2}{y_2 - y_1}$$



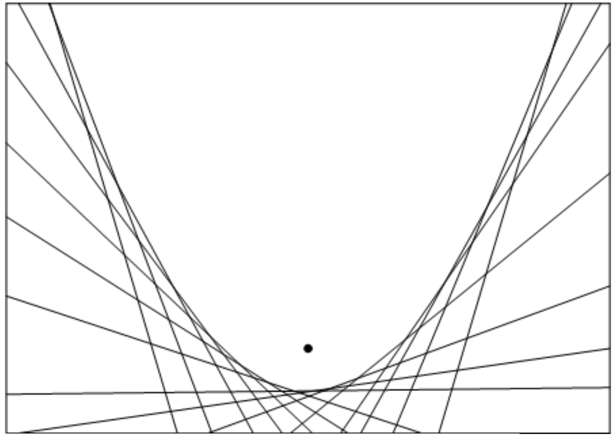
Equation of the parabola

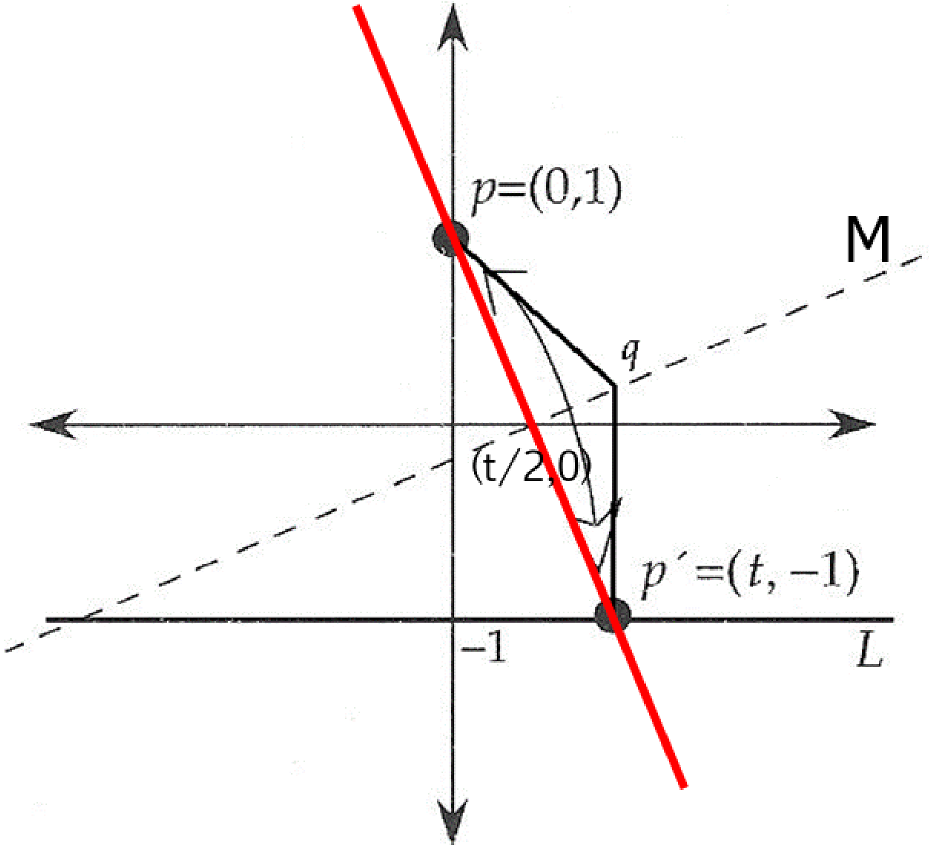
- Fold point $p' = (t, -1)$ to $p = (0,1)$ creates a new crease line M.
- The red line has slope

$$m = \frac{-1-1}{t\ -0} = \frac{-2}{t}$$

- So M has slope

$$\frac{-1}{m} = \frac{t}{2}$$



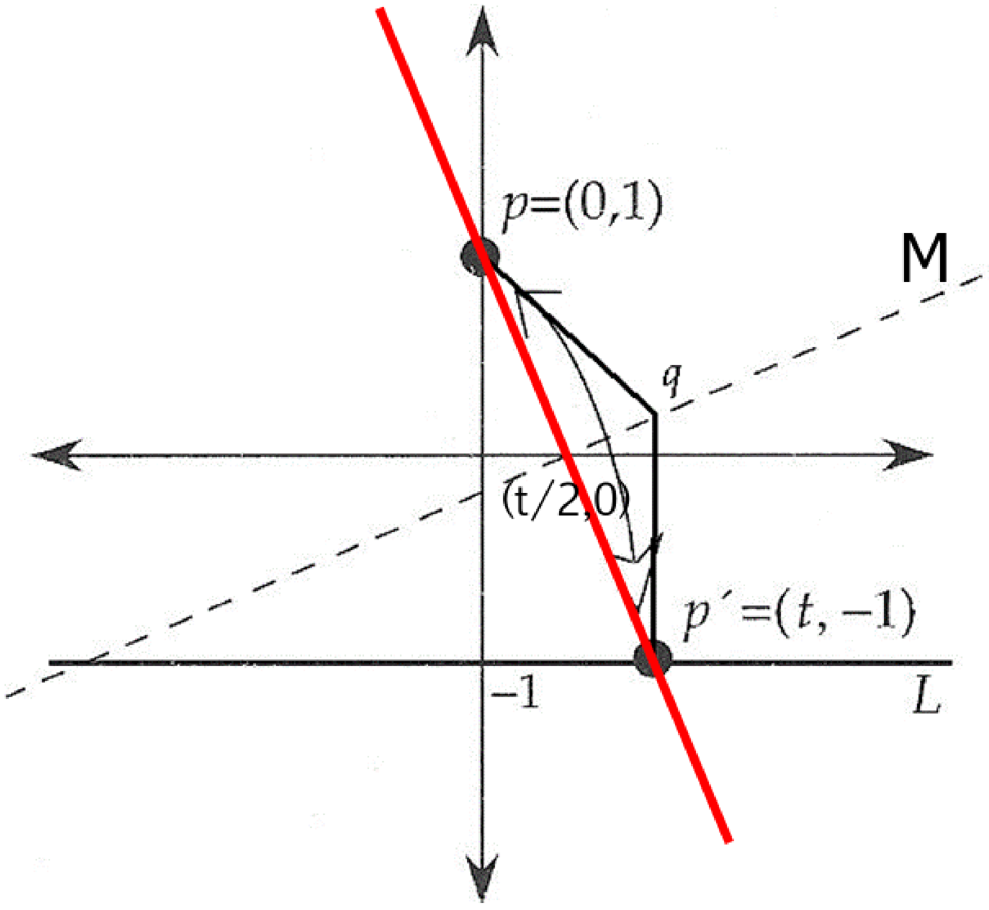


Source: http://www.cutoutfoldup.com/507-fold-a-parabola.php

Equation of the parabola

- M is perpendicular to pp', so M has slope $\frac{-1}{m} = \frac{t}{2}$.
- The mid point of segment pp', point $(\frac{t}{2}, 0)$ lies on M, so $0 = \frac{t}{2}\frac{t}{2} + c$.
- The intercept of M is $c = \frac{-t^2}{2}$, and equation of M is

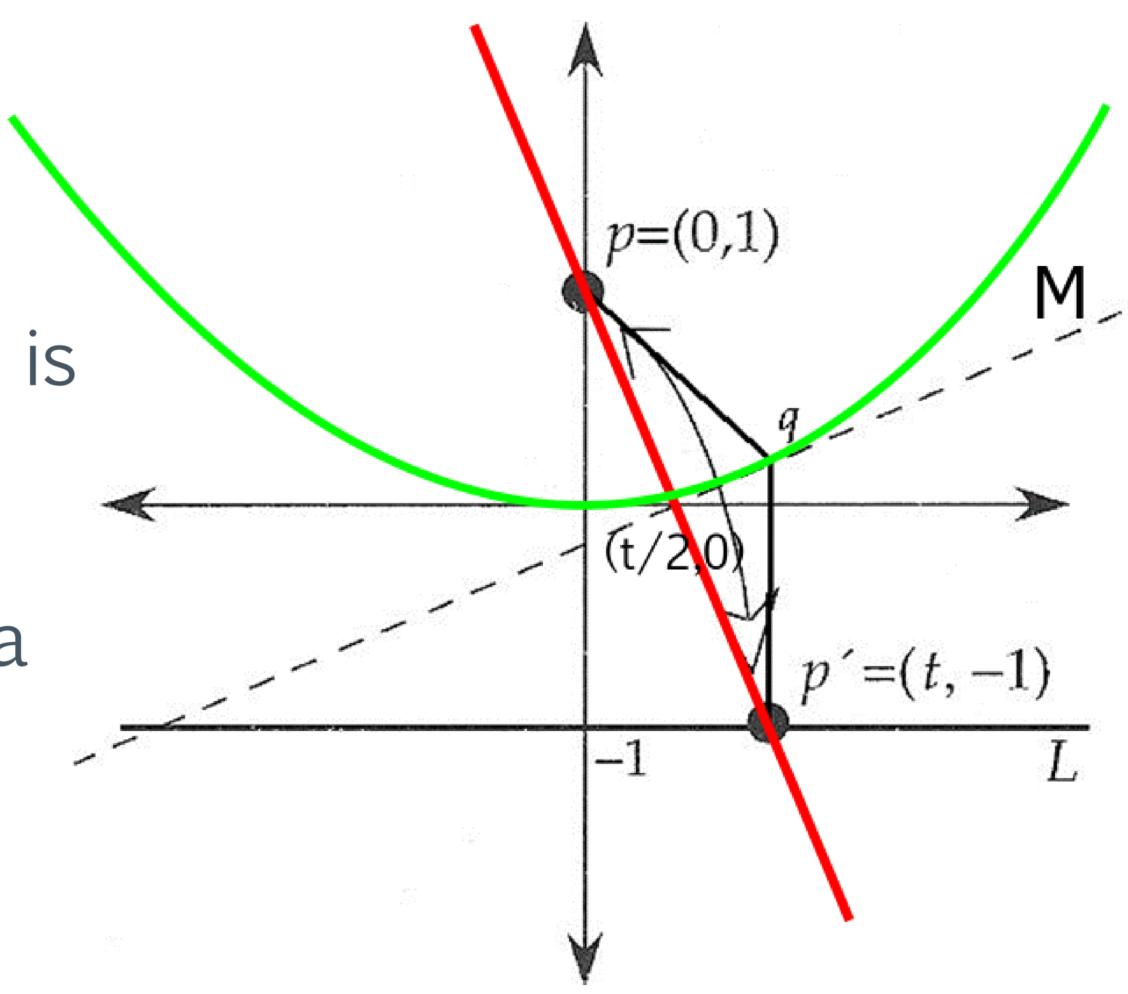
$$y = \frac{t}{2}(x - \frac{t}{2})$$



Source: http://www.cutoutfoldup.com/507-fold-a-parabola.php

Equation of the parabola

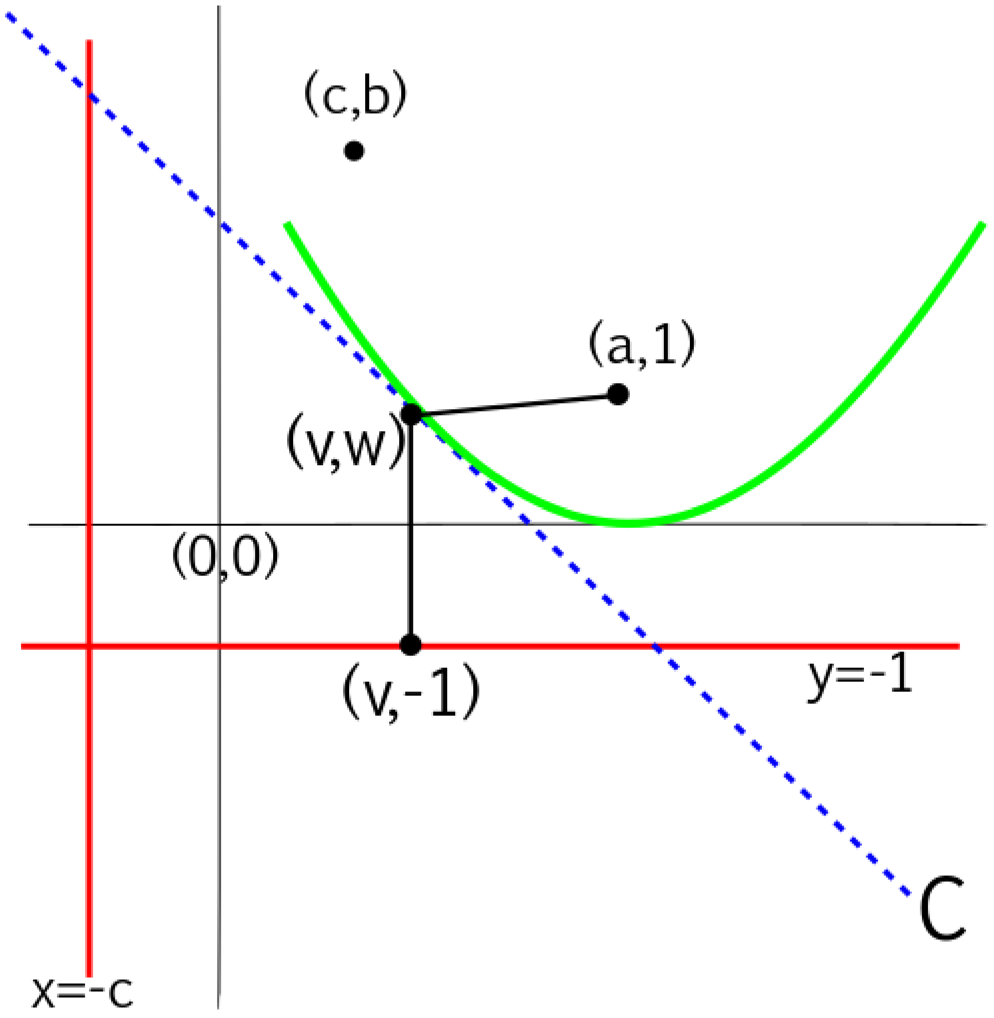
- Equation of M is
$$y = \frac{t}{2}(x - \frac{t}{2})$$
- Point q lies on the parabola is at
$$(t, t^2/4)$$
- So equation for the parabola is
$$y = x^2/4$$



Source: http://www.cutoutfoldup.com/507-fold-a-parabola.php

Explanation

- The crease line C is given by the equation $y = tx + u$.
- The slope is t and the intercept is u.
- We show
  $$t^3 + at^2 + bt + c = 0.$$
- Then t is a solution.



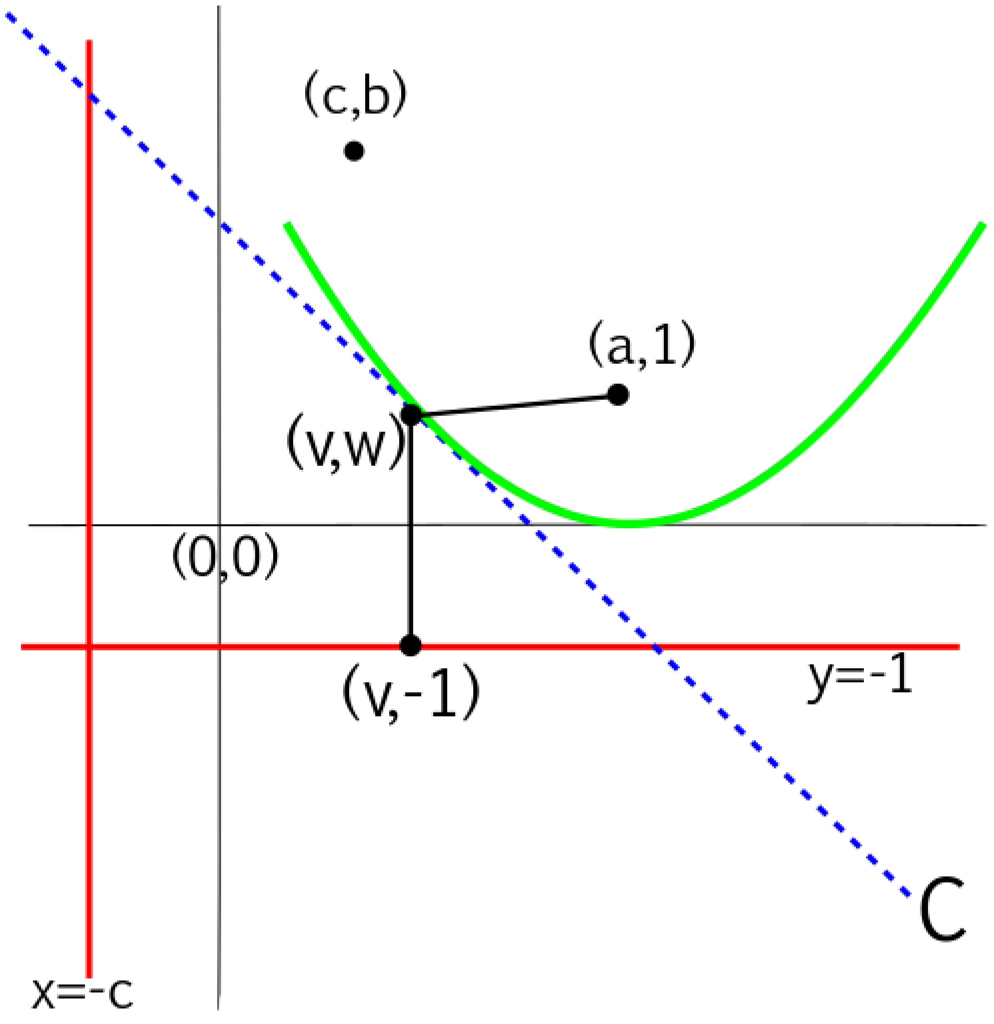
Explanation

- Draw a parabola with focus $p_1 = (a, 1)$ and directrix $L_1 = \{y = -1\}$.

- (v, w) lies on C and also the parabola.

- Equation of parabola is

$$y = \frac{1}{4}(x - a)^2$$



Explanation

- Equation of parabola is

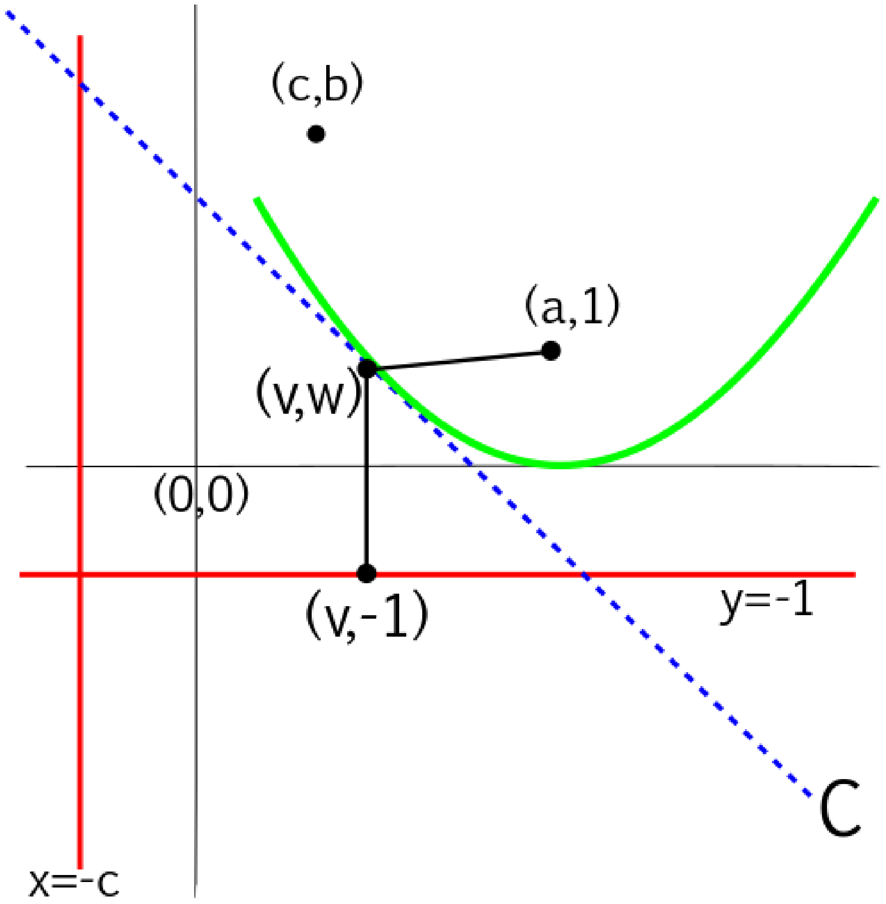
$$y = \frac{1}{4}(x-a)^2$$

- Slope of C at (v, w) is

$$\frac{1}{2}(v-a)$$

- So equation for C is

$$y = \frac{1}{2}(v-a)x + \left(w - \frac{1}{2}(v-a)v\right)$$



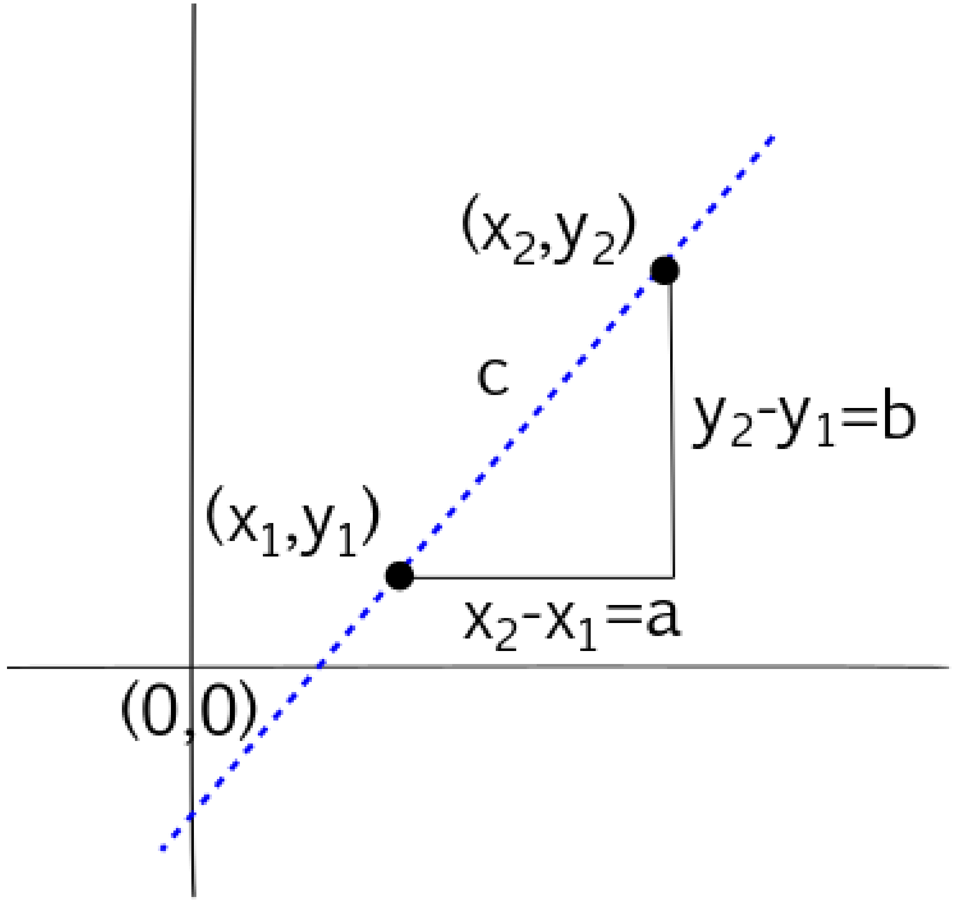
Revision – distance between two points

- Pythagoras' Theorem (畢氏定理)

$$a^2 + b^2 = c^2$$

- Distance between (x_1, y_1) and (x_2, y_2) is

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Explanation

- Distance$\big((v,w),(v,-1)\big) =$ Distance$\big((v,w),(a,1)\big)$, so

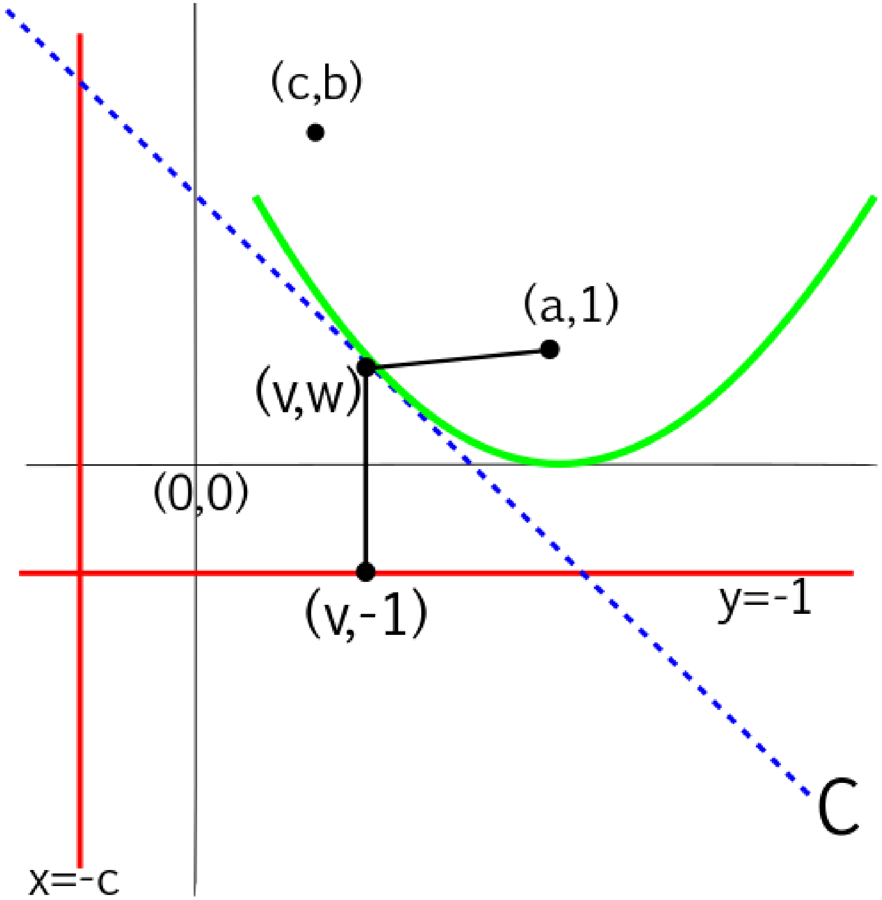
$$\sqrt{(a-v)^2+(1-w)^2} = \sqrt{(w+1)^2}$$

$$or$$

$$(a-v)^2 = 4w$$

- Set $t = \frac{1}{2}(v-a)$, then $w = t^2$ and equation for C is

$$y = tx + u,$$
$$u = -t^2 - at$$

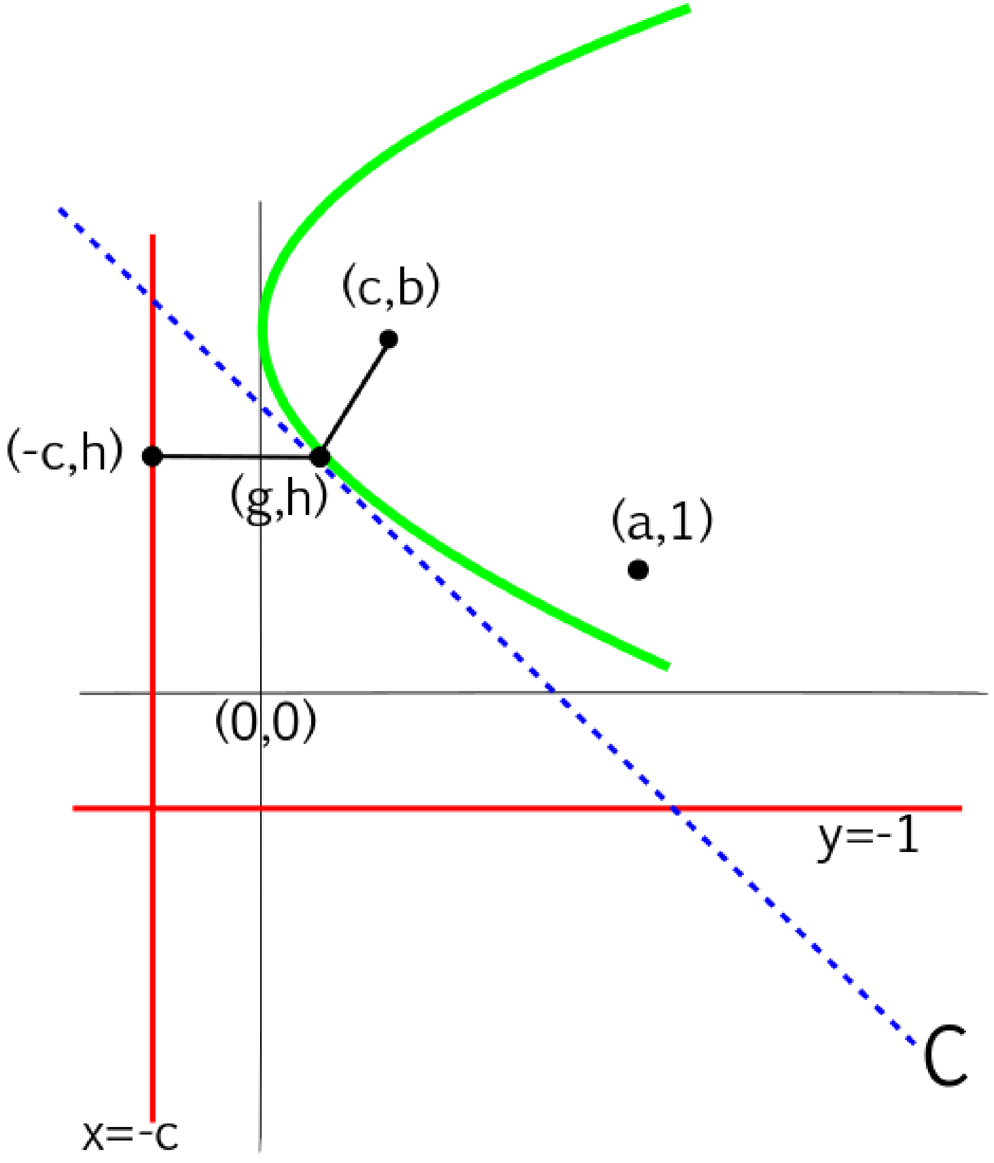


Explanation

- Draw another parabola with focus $p_2 = (c, b)$ and directrix $L_2 = \{x = -c\}$.

- (g, h) lies on C and also on the parabola.

- Equation of parabola is
$$cx = \frac{1}{4}(y - g)^2$$



Explanation

- Equation of parabola is

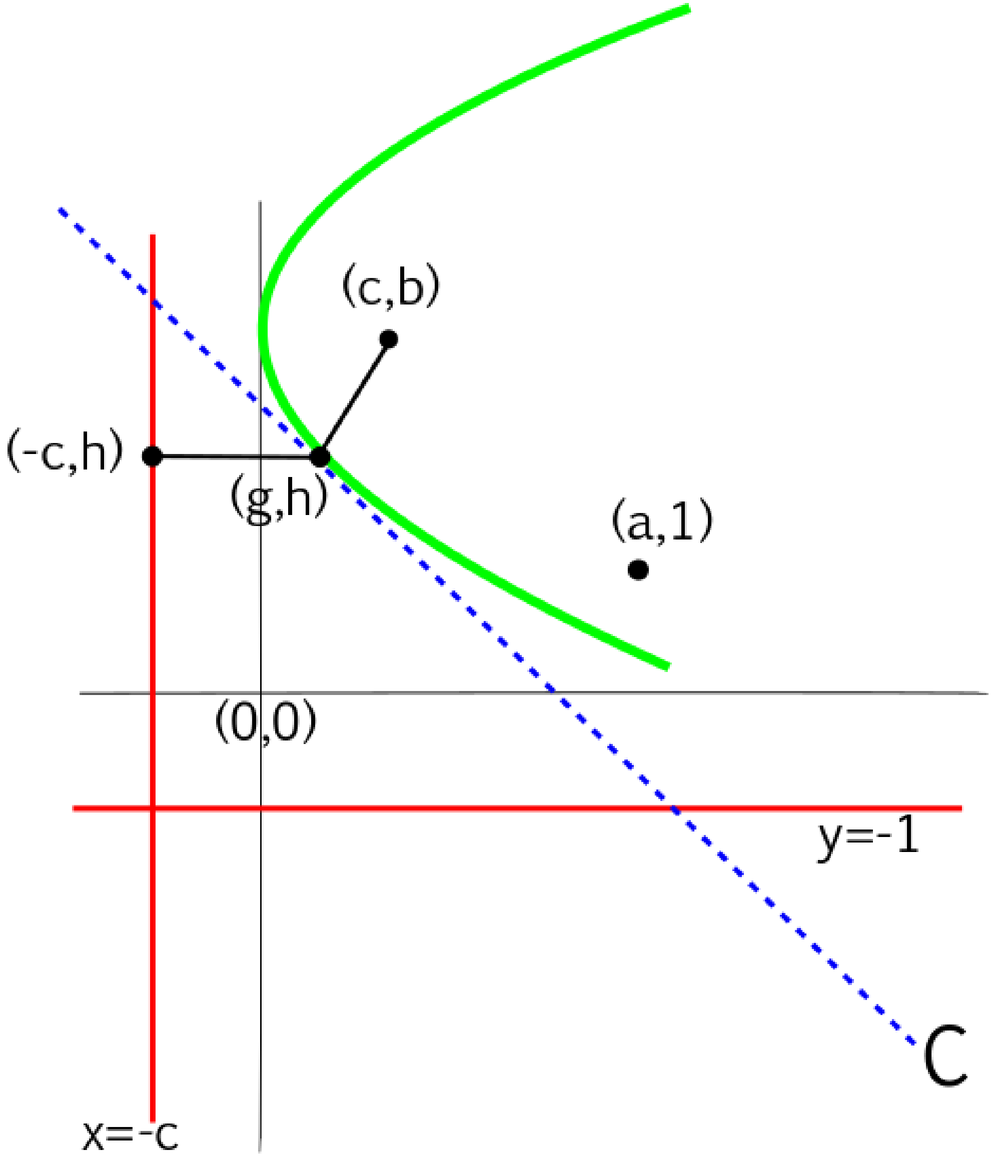
$$cx = \frac{1}{4}(y-g)^2$$

- Slope of C at (g,h) is

$$\frac{2c}{(h-b)}$$

- So equation for C is

$$y = \frac{2c}{(h-b)}x + \left(h - \frac{2c}{h-b}g\right)$$



Explanation

- Distance$\big((g,h),(-c,h)\big)$ = Distance$\big((g,h),(c,b)\big)$,

$$\sqrt{(g-c)^2+(h-b)^2} = \sqrt{(g+c)^2}$$

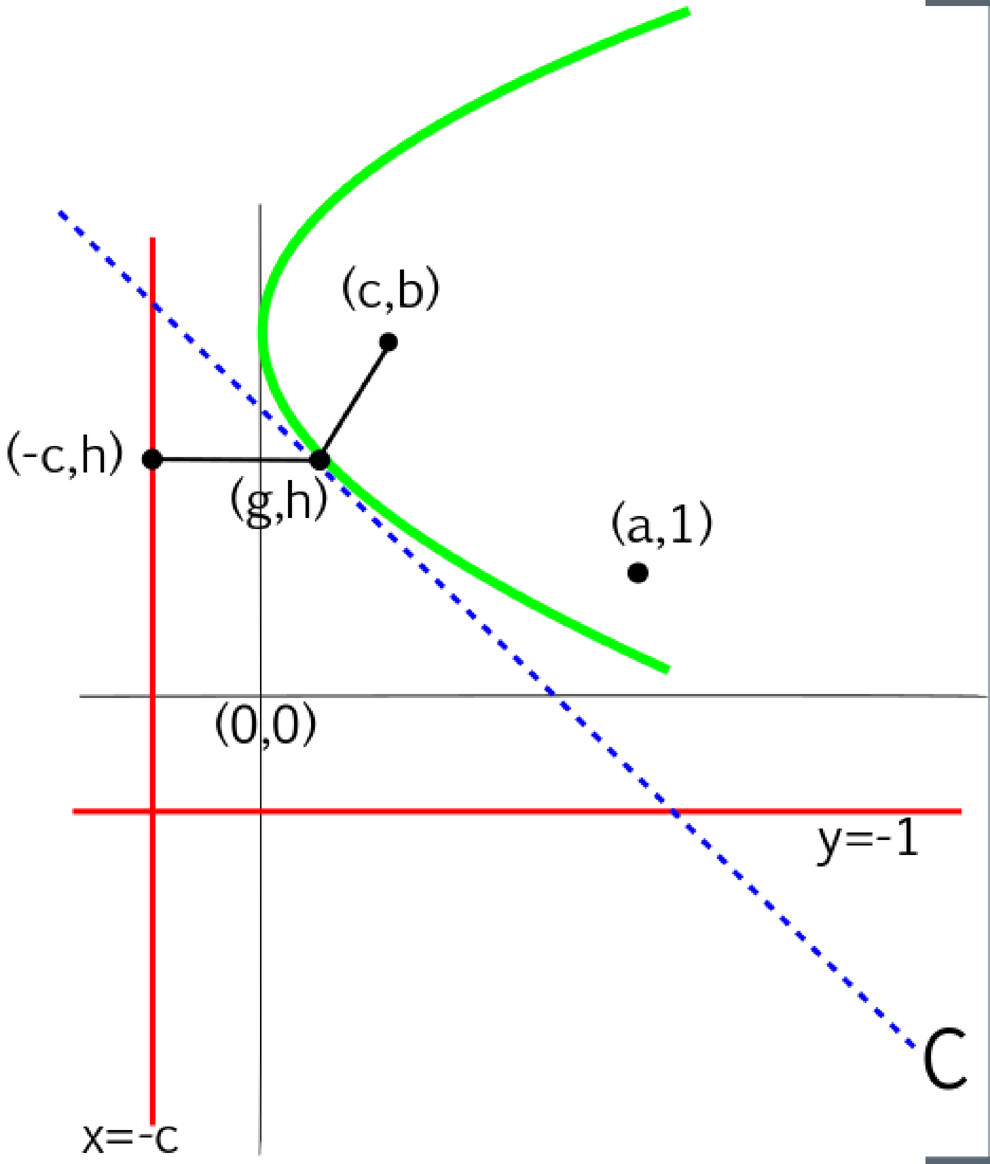
$$or\ (h-b)^2 = 4gc\ .$$

- So equation for C is

$$y = \frac{2c}{(h-b)}x + \left(h - \frac{2c}{h-b}g\right)$$

simplifies to

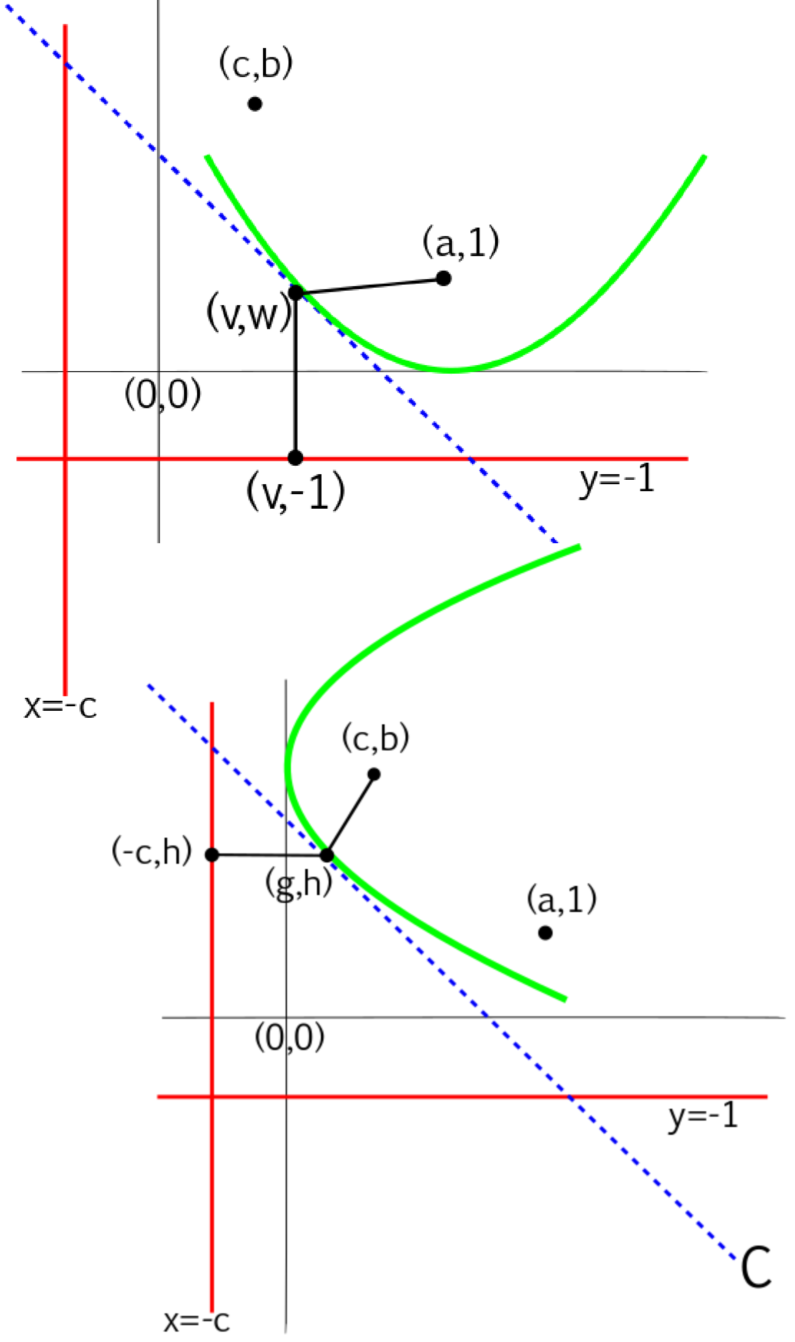
$$y = tx + u, \quad t = \frac{2c}{(h-b)}, \quad u = b + \frac{c}{t}.$$



Explanation

- First picture:
  $y = tx + u, \quad t = \frac{1}{2}(v - a) \quad u = -t^2 - at$
- Second picture:
  $y = tx + u, \quad t = \frac{2c}{(h-b)}, \quad u = b + \frac{c}{t}$
- Intercept u is the same, so

  $-t^2 - at = b + \frac{c}{t}$ or $t^3 + at^2 + bt + c = 0$

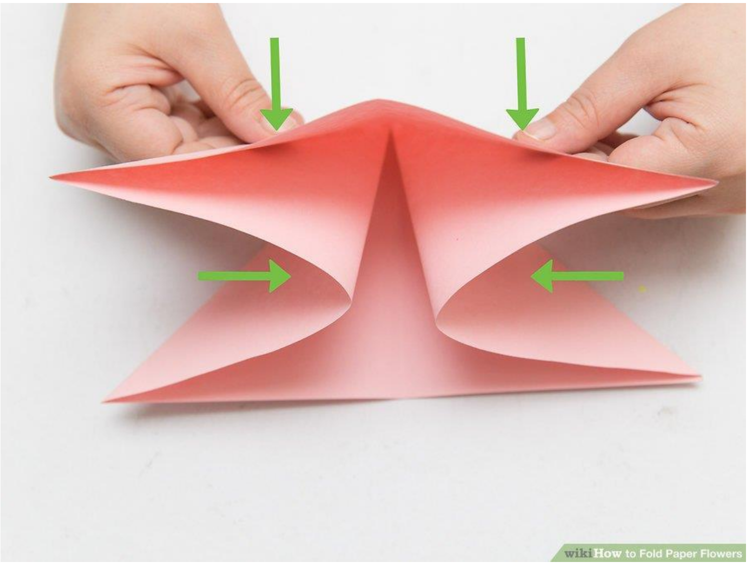


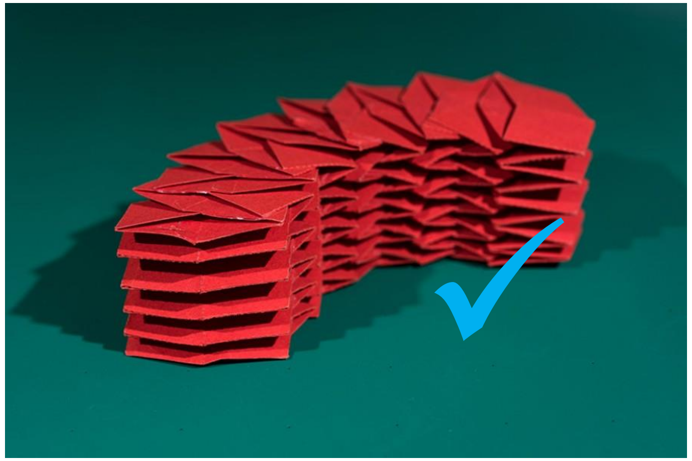
Mathematics for (flat) Origami

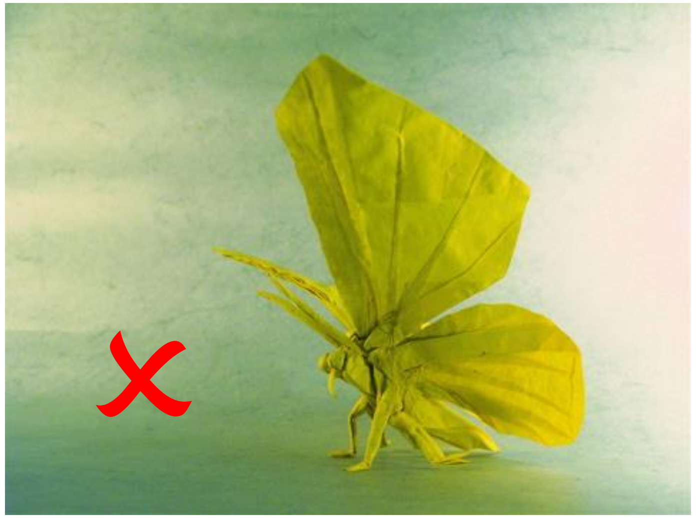
Flat origami

› An origami is **flat foldable** (可以被摺平) if it can be compressed without making new creases.

› Useful if you want to put it into your pocket.



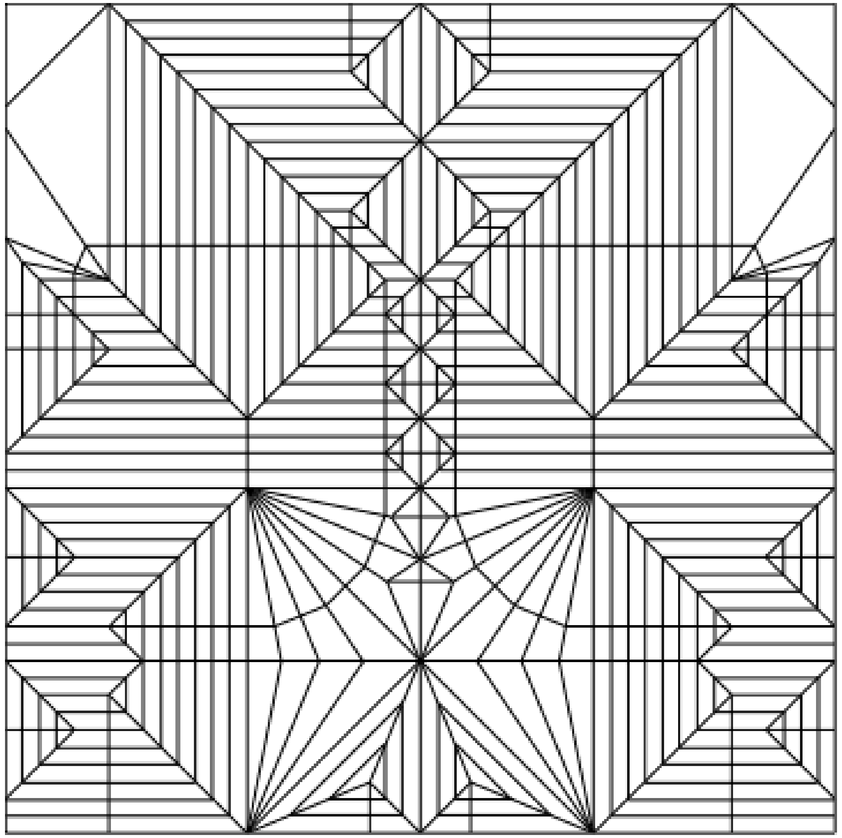




Source: https://www.wikihow.com/Fold-Paper-Flowers
https://www.ce.gatech.edu/news/researchers-develop-new-zippered-origami-tubes-fold-flat-deploy-easily-and-still-hold
http://jasonku.mit.edu/butterfly1.html

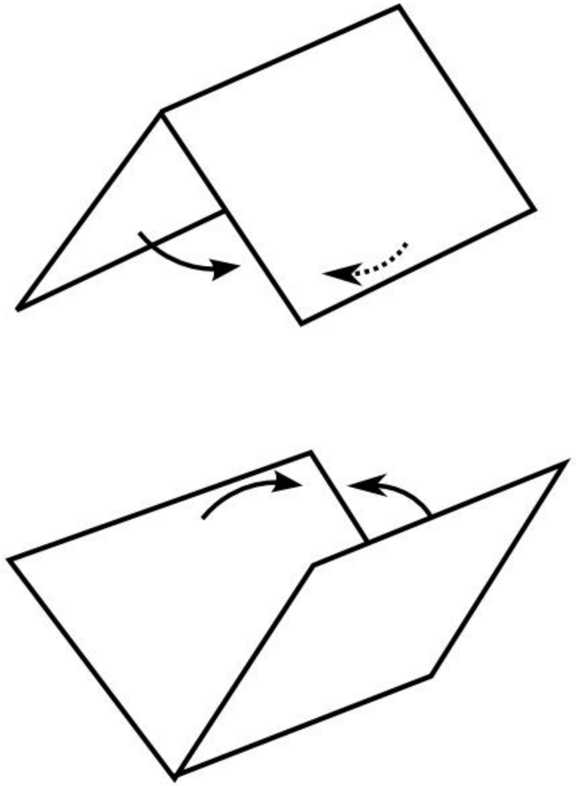
Is it flat?

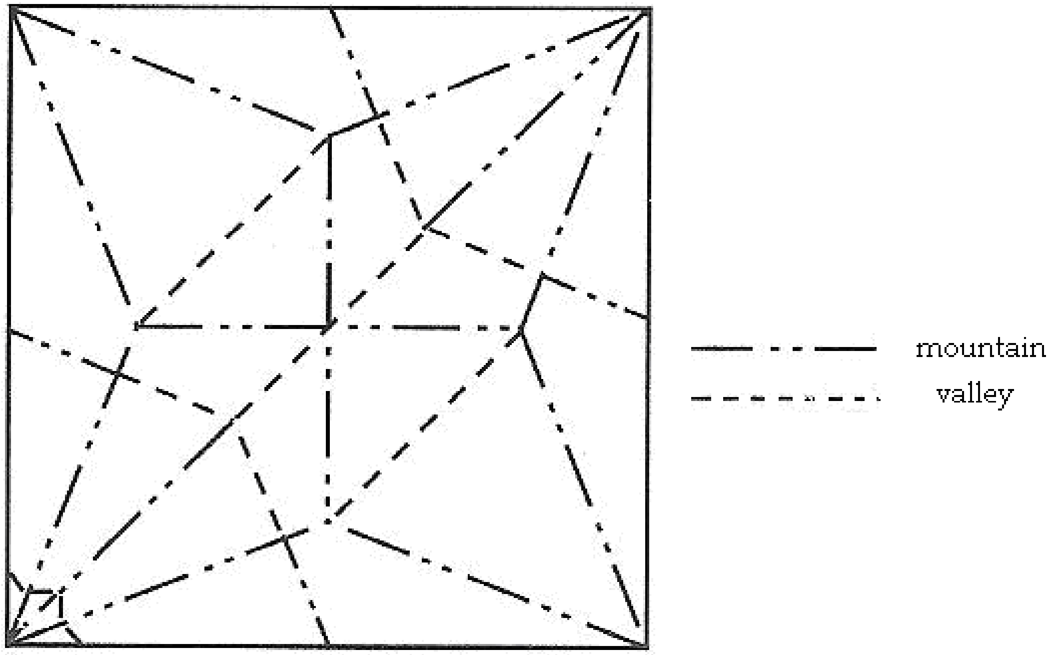
- Can we judge whether an origami is **<u>flat</u>** just by looking at the crease pattern (摺痕)?
- We use some mathematical ideas and tools to investigate.



Mountains and Valleys

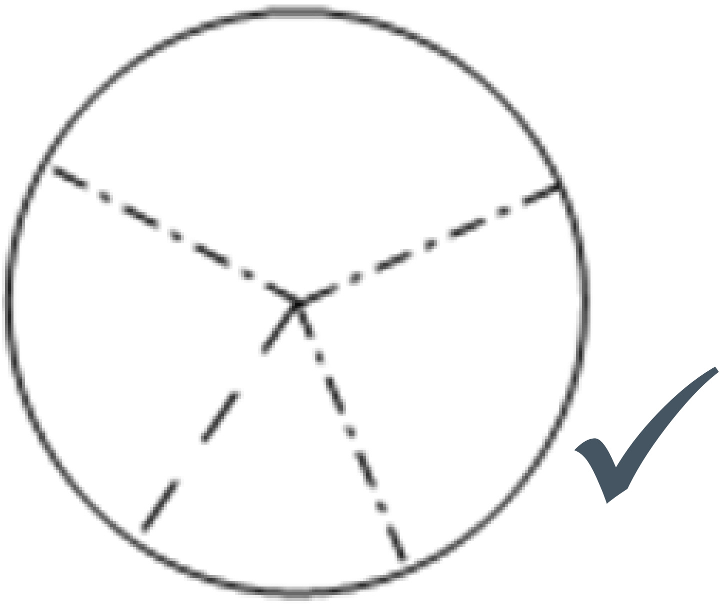
› In origami, there are only two ways to make a fold: Mountain (山) or Valley (谷)

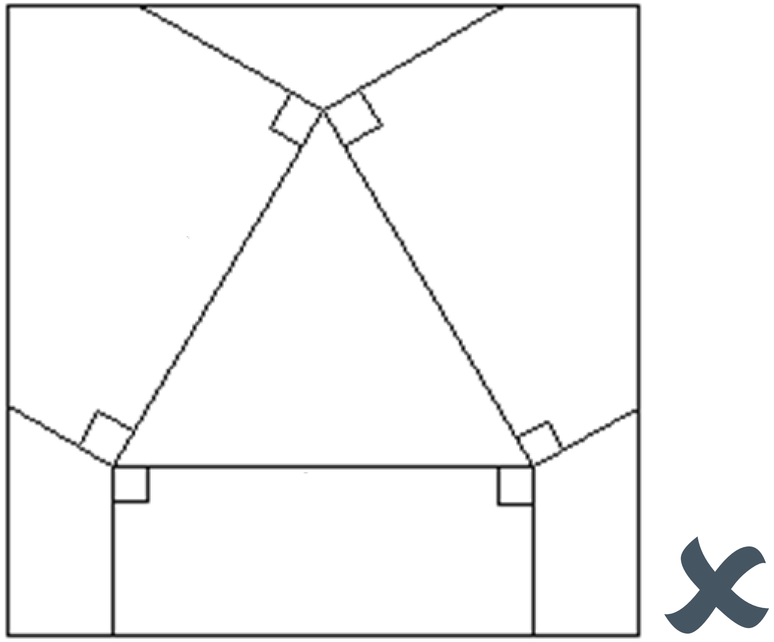




Single vertex origami

- A vertex (頂點) is any point inside the paper where two or more crease lines (摺痕線) meet (交叉).
- A single vertex (單頂點) origami only has one vertex.
- We study when is a single vertex origami flat foldable





Maekawa's Theorem

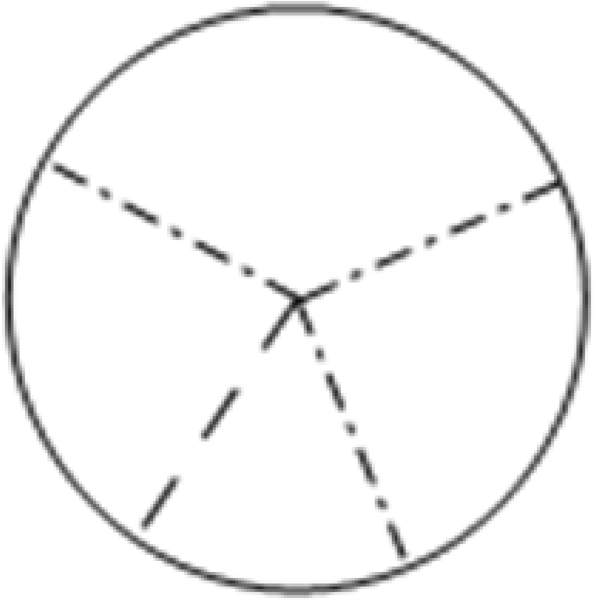
- MAEKAWA Jun (前川　淳)– Japanese software engineer, mathematician, origami artist.
- If an origami is flat vertex fold, then

$$M - V = 2 \quad or \quad M - V = -2$$

- **Meaning ?**
  (a) The difference between mountain folds and valley folds is always 2.

  (b)Total number of creases (摺痕總數) must be **even** (雙數)

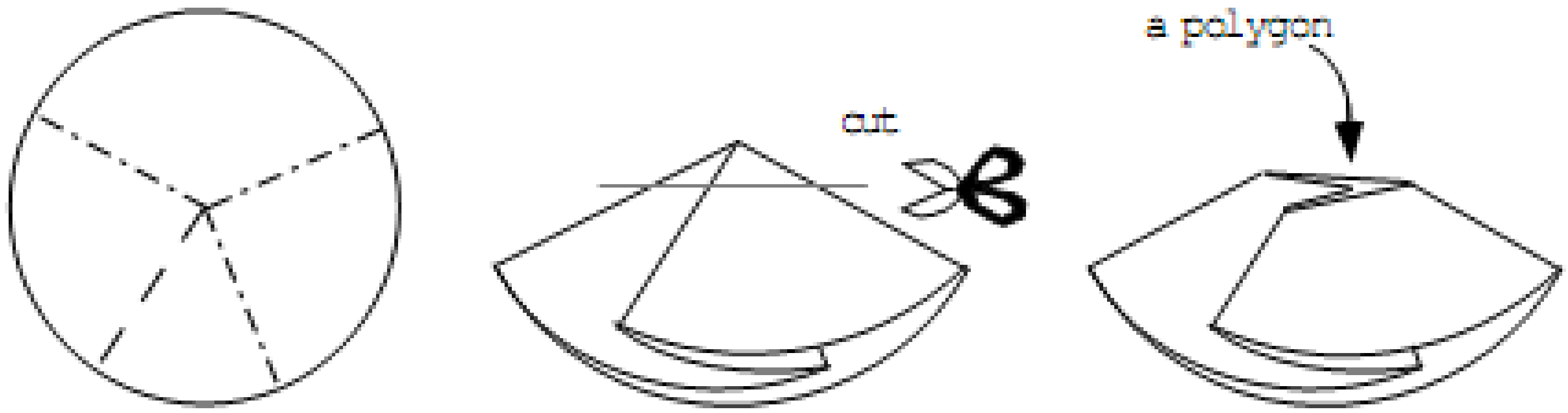
$$M + V = 2V + 2\ (or\ 2V - 2)$$



Source: https://en.wikipedia.org/wiki/Jun_Maekawa
https://origamicaravan.org/archives/2011-artists/jun-maekawa/

The proof

- If origami is flat foldable at one vertex, then cutting off the vertex after folding gives a polygon (多邊形).

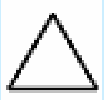
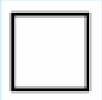
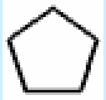
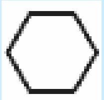
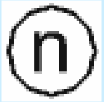


- If the total number of creases is n, then

$$M + V = n.$$

Revision – angle sum (角度總和) in a polygon

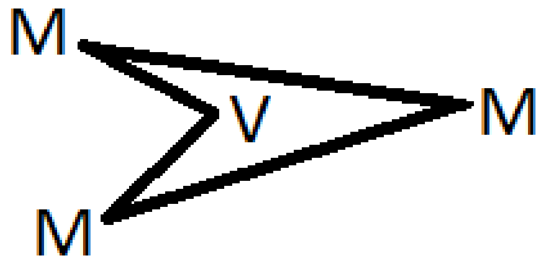
- Sum of all angles in any triangle (三角形) is always 180°.
- There are 2 triangles in a quadrilateral (四邊形), so the sum of all angles is always 360°.
- For a polygon of n sides, we can fit $n-2$ triangles, so the sum of all angles is $(n-2)*180°$.

Sides	Sum of Interior Angles	Shape
3	180°	
4	360°	
5	540°	
6	720°	
...	..	...
n	(**n**-2) × 180°	

Source: https://www.mathsisfun.com/geometry/interior-angles-polygons.html

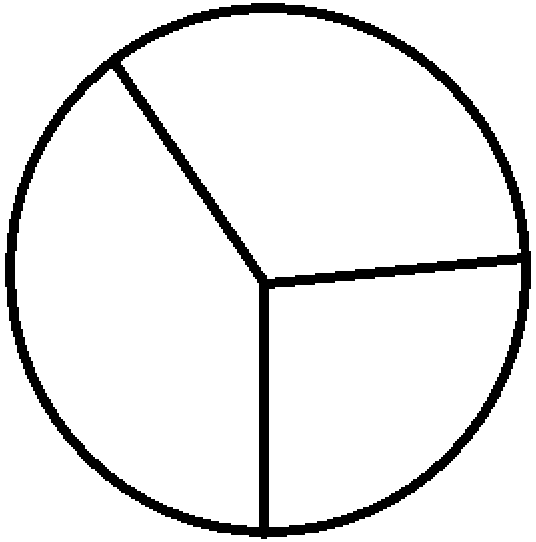
The proof

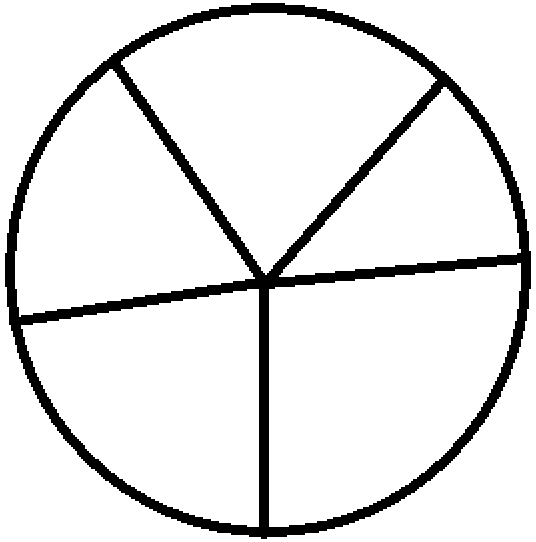
- Label each valley fold V with $360°$ angle and each mountain fold M with $0°$ angle.
- The angle sum is
$$360° * V + 0° * M = 360° * V$$
- But angle sum of n-sided polygon is
$$(n-2) * 180°$$
- So
$$360° * V = (M + V - 2) * 180°$$
$$\text{or} \quad 2V = M + V - 2$$
$$\text{or} \quad \mathrm{M} - \mathrm{V} = 2$$
- Reverse labelling (V with $0°$ and M with $360°$) gives $M - V = -2$.



Question??

› If there is an odd number (單數) of creases (摺痕線) to a vertex origami (單頂點摺紙), can it be folded flat?





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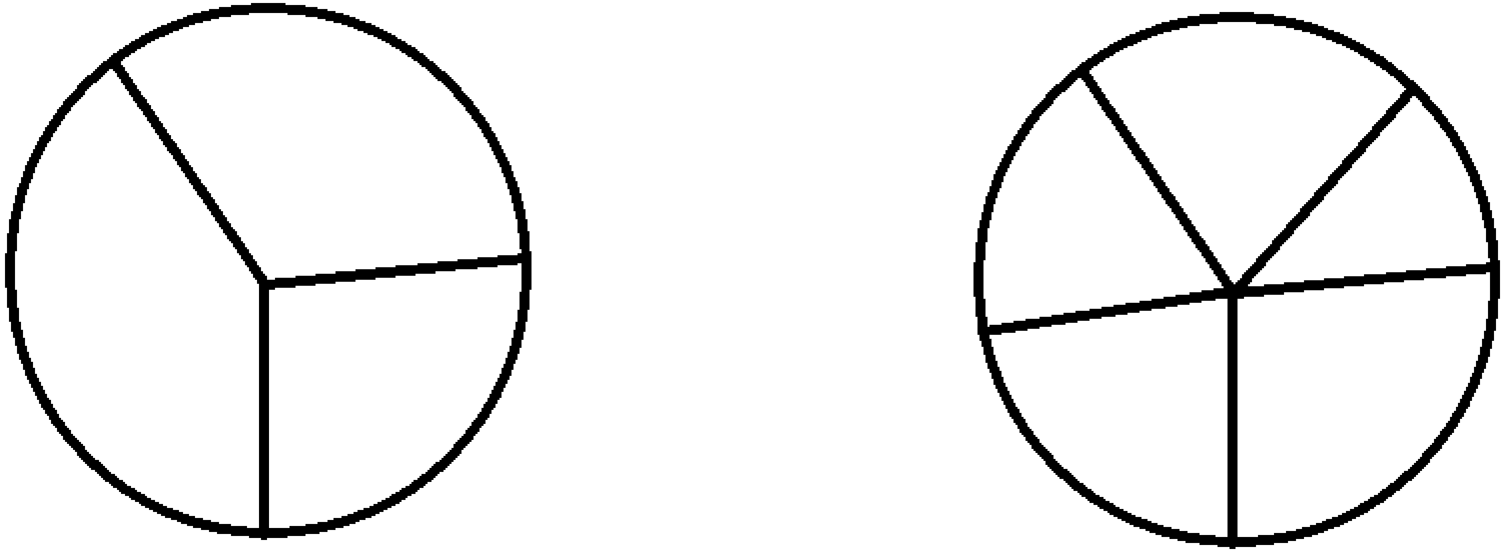


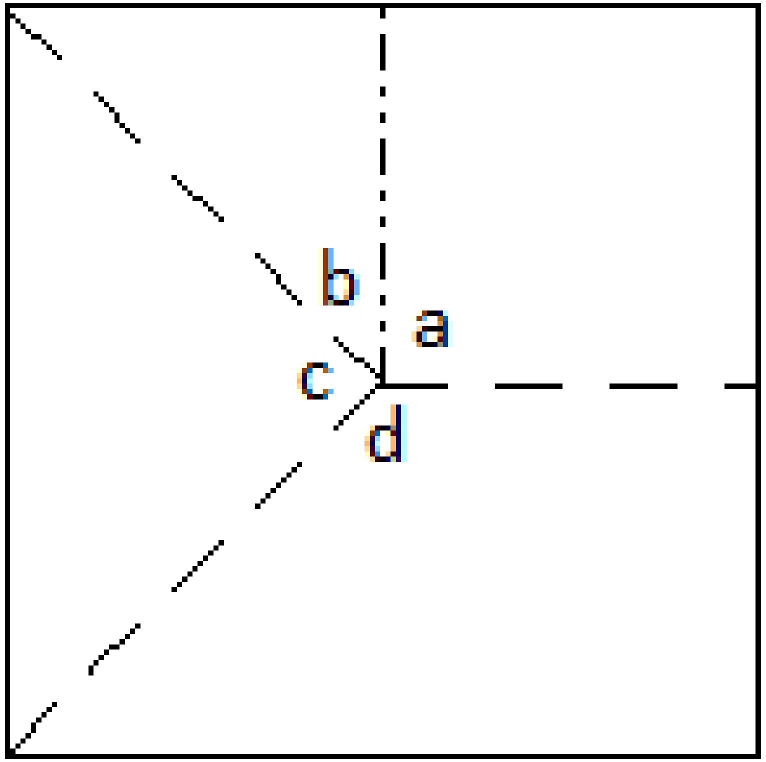
- Answer: **No!** Because of Maekawa's theorem.

Kawasaki's Theorem

- KAWASAKI Toshikazu (川崎 敏和) – Japanese paperfolder, famous for the Kawasaki Rose.
- If the **alternating sum of consecutive angles (交替連續總和角度) = 0**°, e.g.

$$a - b + c - d = 0°$$

then the origami is flat foldable.



Source: https://en.wikipedia.org/wiki/Toshikazu_Kawasaki
https://origamicaravan.org/archives/2011-artists/toshikazu-kawasaki/

Application of Kawasaki's Theorem

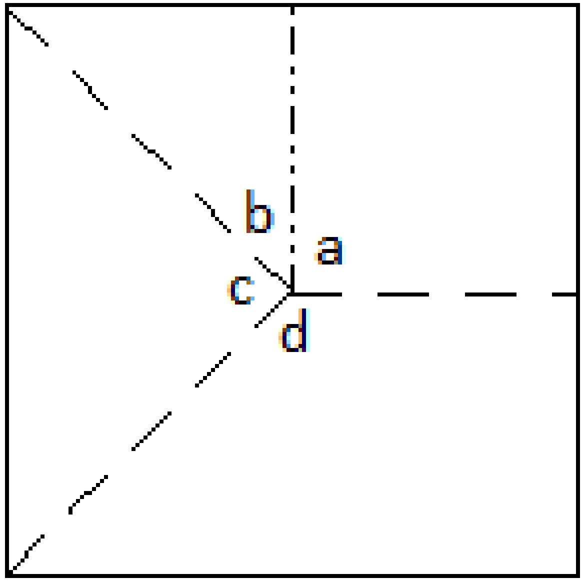
- In the example,

$$a = 90°, b = 45°, c = 90°, d = 135°$$

- The alternating sum is

$$a - b + c - d = 90° - 45° + 90° - 135° = 0°$$

- So Kawasaki's theorem tells us it is flat foldable.



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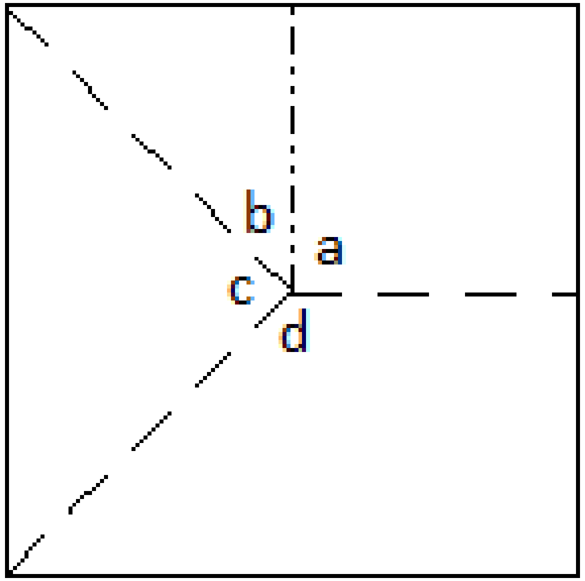

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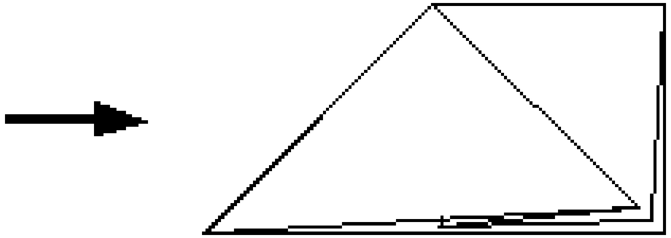
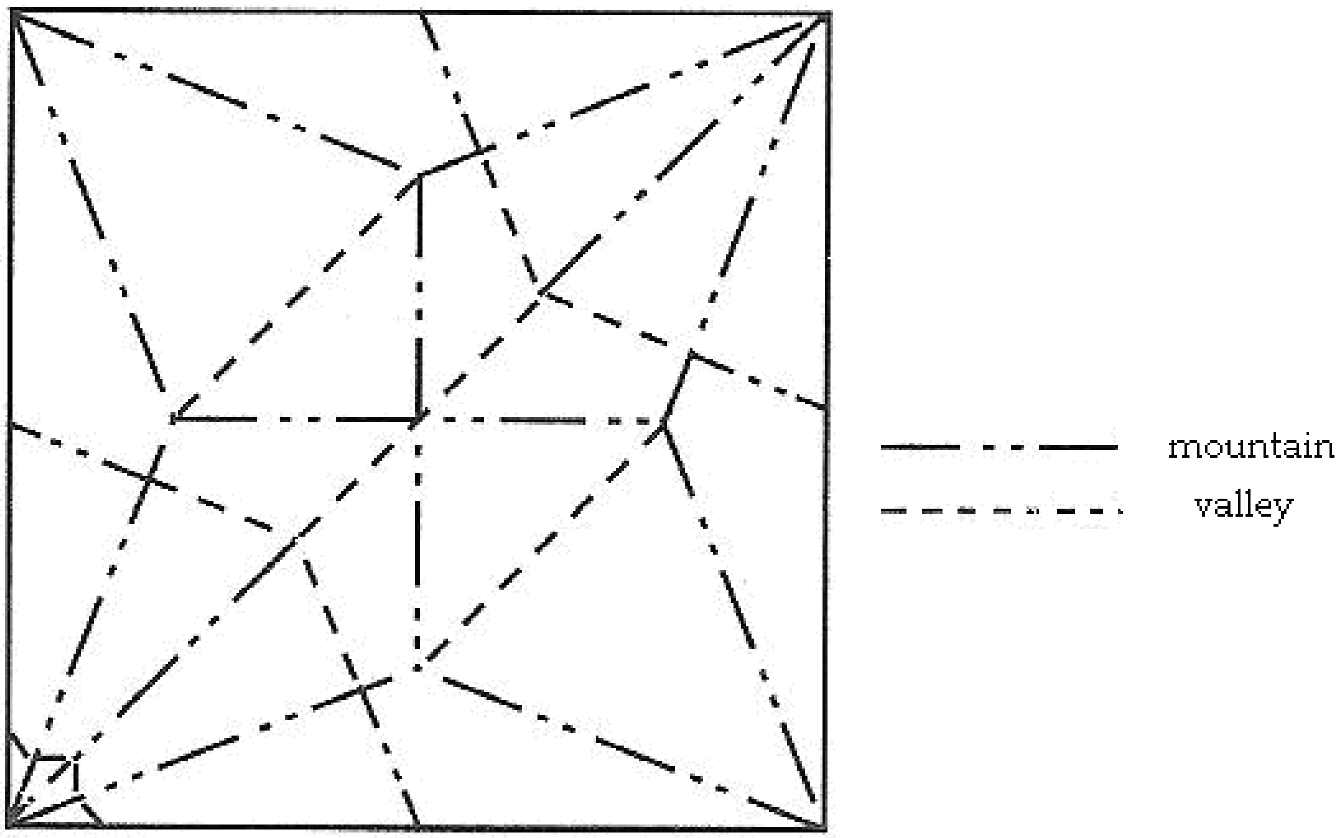


What about with more vertices (多個頂點)?

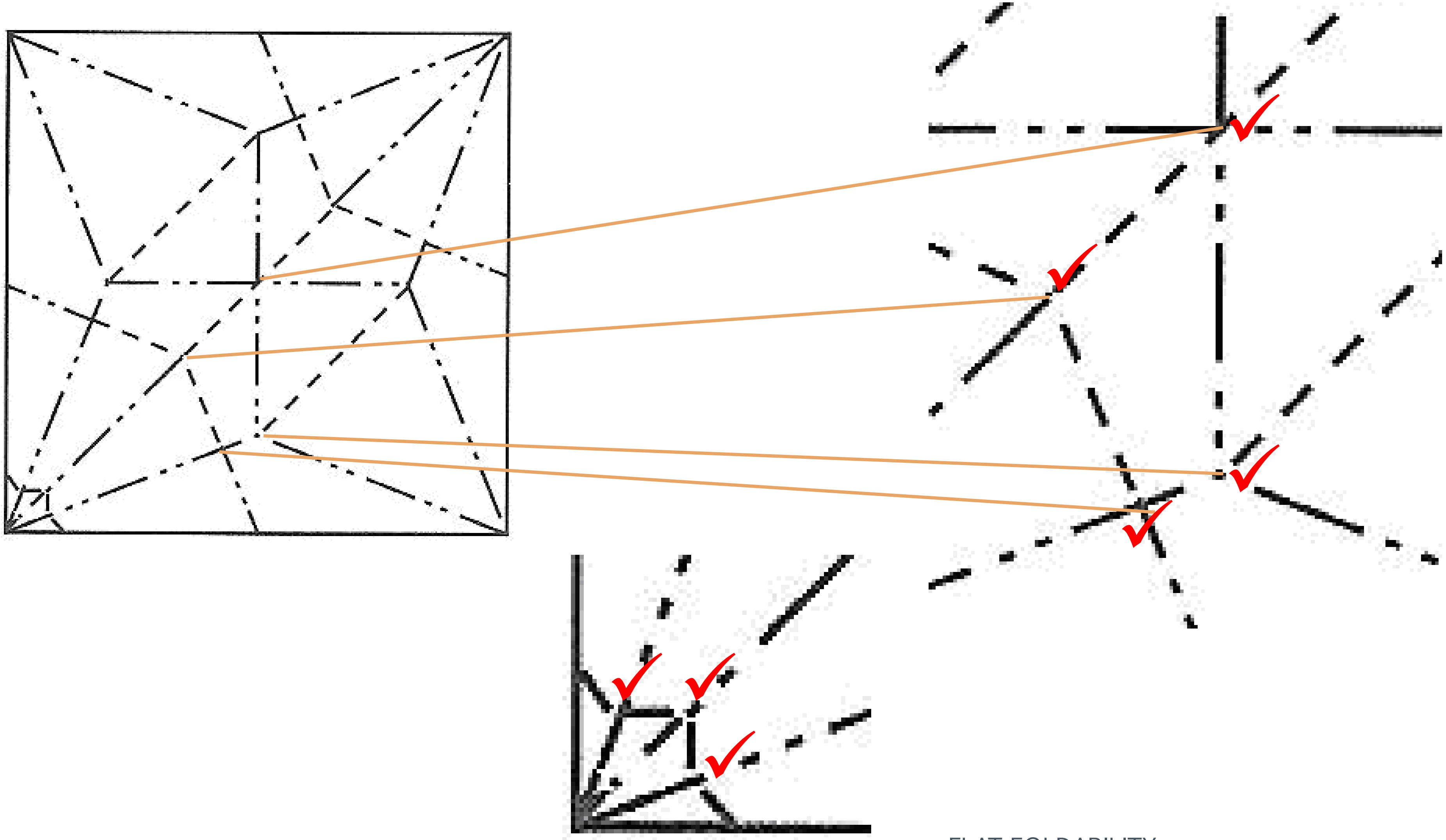
- Both Maekawa's theorem and Kawasaki's theorem are for single vertex folds (單頂點摺紙).
- But these are not very interesting origami.
- Is there a version for crease patterns with more than one vertex?

Positive example (正例子)

› Investigate if the crease pattern gives a flat foldable origami.

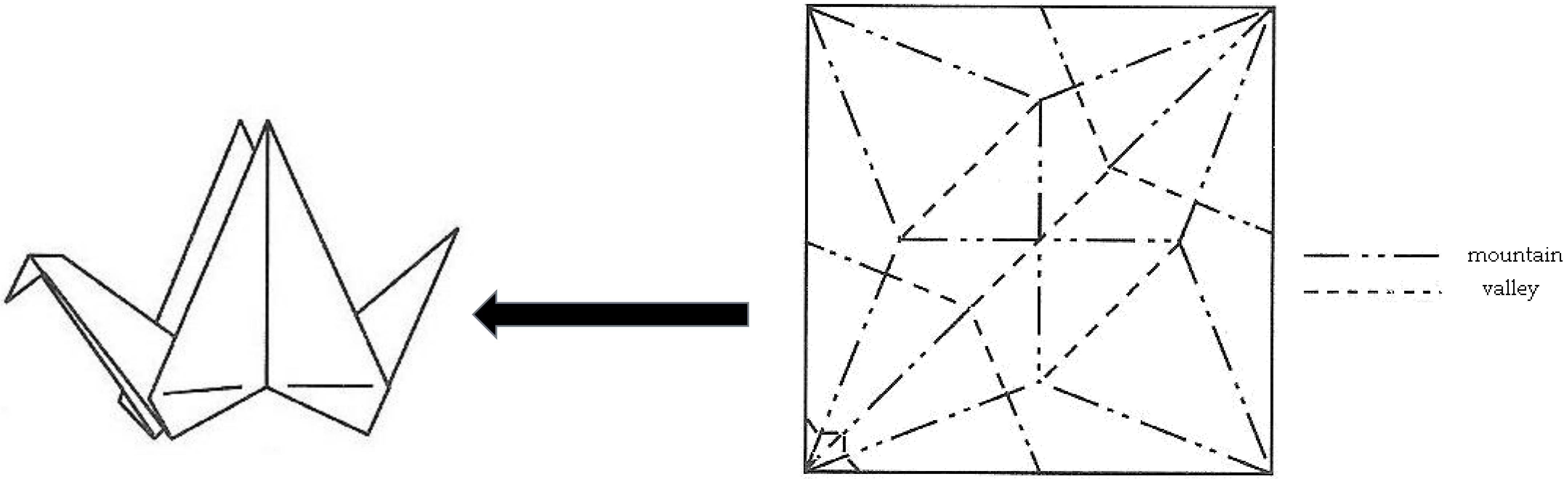


Positive example (正例子)



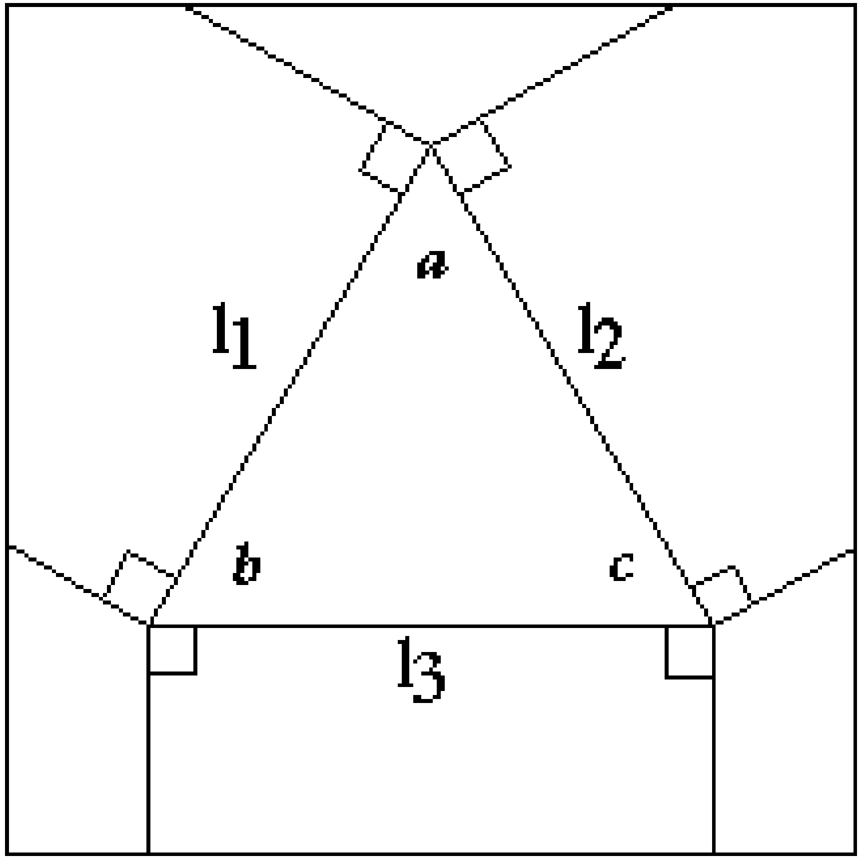
Positive example (正例子)

› It is flat foldable as it is the crease pattern for a crane.



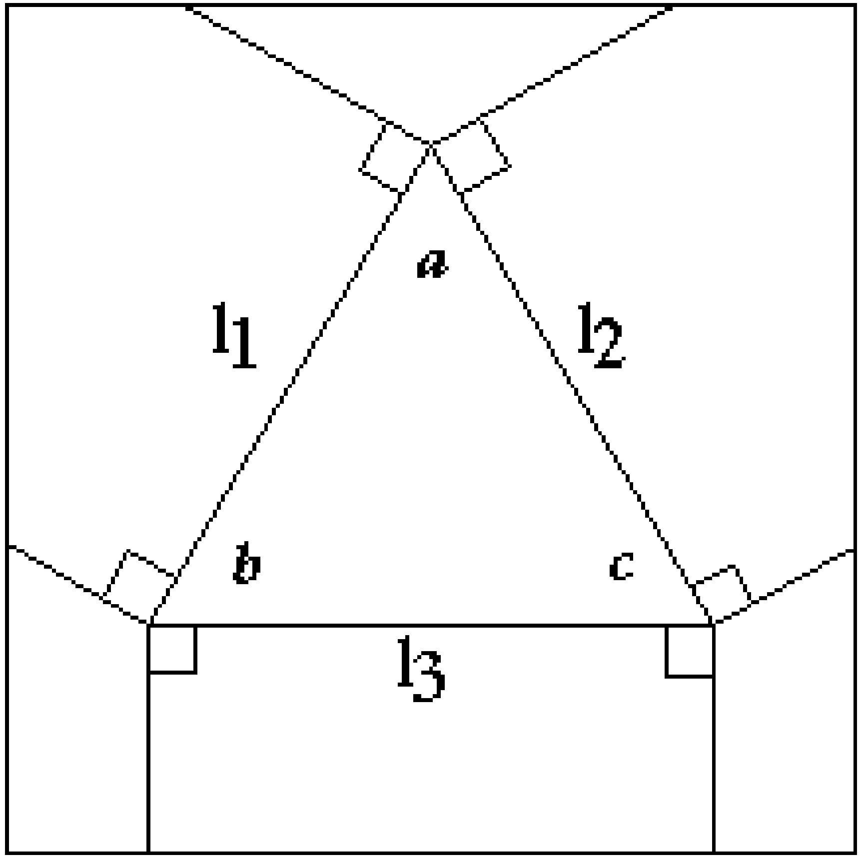
Negative example (負例子)

- Angles $a = b = c = 60°$.
- Each vertex has alternating sum
  $90° - 120° + 90° - 60° = 0°$
- Kawasaki's theorem says the origami can be flat foldable at any one vertex.
- What about the instruction for each crease?



Negative example (負例子)

- If l_1, l_2 are both the same (e.g. mountain), then the origami cannot be fold flat – otherwise paper intersect itself (自我相交).
- So l_1 must be folded differently to l_2.
- E.g. l_1 is a mountain, l_2 is a valley.



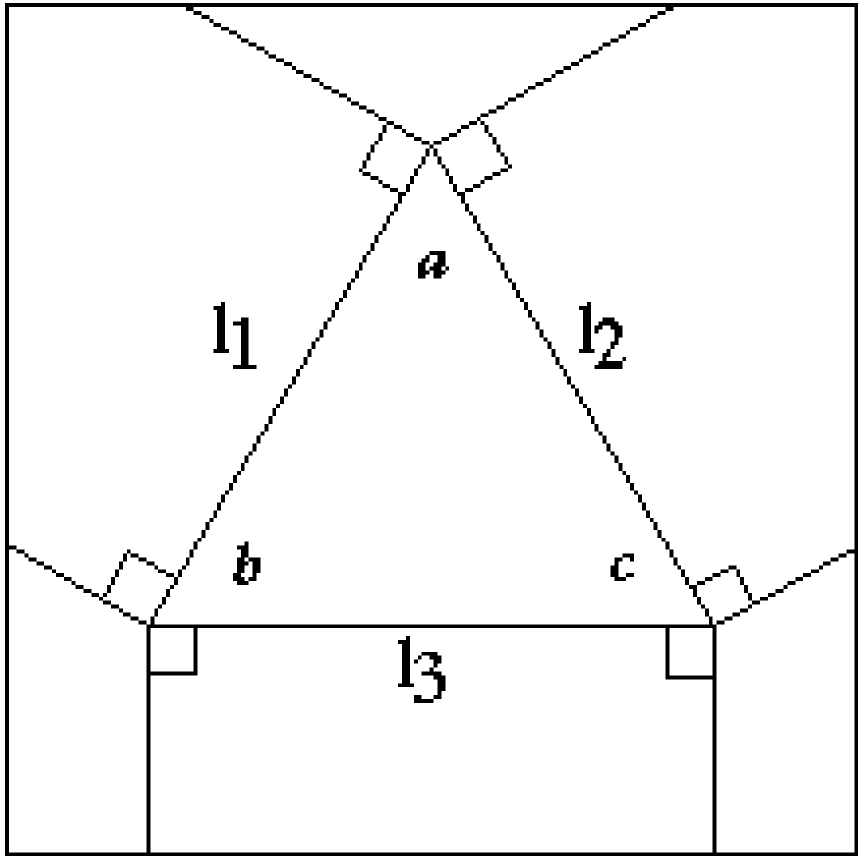
Negative example (負例子)

› l_1 is a mountain, l_2 is a valley.

› Same idea: l_2 and l_3 cannot be the same, so l_3 is a mountain.

› Now, we have a contradiction (矛盾) with l_1 and l_3.

› So the origami **cannot be folded flat!**



What's wrong?

- Kawasaki's theorem is for crease pattern with a single vertex.
- For multiple vertices, Kawasaki's condition is not sufficient. We saw one negative example !'
- Therefore, the answer to flat foldability (for multiple vertices) is still <u>unknown</u>. This is an <u>open</u> problem in mathematical origami.

Thank you for your attention!