Binomial Theorem Exercise (part 2) Answer

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1 Elementary questions

Q1) Consider

$$(1+x)^9 = \sum_{k=0}^{9} \binom{9}{k} \times x^k$$

Put

$$x = 1$$

, we get

$$(1+1)^9 = \sum_{k=0}^{9} \binom{9}{k} \times 1^9 = \sum_{k=0}^{9} \binom{9}{k}$$

Hence,

$$\sum_{k=0}^{9} \binom{9}{k} = 2^9$$

Q2) Consider

$$(1+x)^{4n} = \sum_{k=0}^{4n} \binom{4n}{k} \times x^k$$

Put

$$x = -1$$

and

$$x = 1$$

, we get

$$\binom{4n}{0}+\binom{4n}{2}+\binom{4n}{4}+\binom{4n}{6}+...+\binom{4n}{4n-2}+\binom{4n}{4n} = \binom{4n}{1}+\binom{4n}{3}+\binom{4n}{5}+\binom{4n}{7}+...+\binom{4n}{4n-3}+\binom{4n}{4n-1}$$

and the sum of left hand side and right hand side are 2^{4n}, so the answer is

$$2^{4n} \div 2 = 2^{4n-1}$$

Q3) Evaluate

$$\binom{20}{1} - \binom{20}{3} + \binom{20}{5} - \binom{20}{7} + ... + \binom{20}{17} - \binom{20}{19}$$

Consider

$$(1+x)^{20} = \sum_{k=0}^{20} \binom{20}{k} \times x^k$$

Put $x = i$, we get

$$(1+i)^{20} = \sum_{k=0}^{10} (-1)^k \binom{20}{2k} + i \times \sum_{k=0}^{10} (-1)^k \binom{20}{2k+1}$$

Note that

$$\sum_{k=0}^{10} (-1)^k \binom{20}{2k+1}$$

is the sum we want to find.

Hence, we need to find the imaginary part of

$$(1+i)^{20}$$

.

$$(1+i)^2 = (1+2i+i^2) = 2i$$

Therefore,

$$(1+i)^{20} = (2i)^{10} = -2^{10}$$

Q4) Evaluate

$$\binom{4n+1}{1} - \binom{4n+1}{3} + \binom{4n+1}{5} - \binom{4n+1}{7} + ... - \binom{4n+1}{4n-1} + \binom{4n+1}{4n+1}$$

and express your answer in terms of n
Consider

$$(1+x)^{4n+1} = \sum_{k=0}^{4n+1} \binom{4n+1}{k} \times x^k$$

Put $x = i$, we get

$$(1+i)^{4n+1} = \sum_{k=0}^{2n} (-1)^k \binom{4n+1}{2k} + i \times \sum_{k=0}^{2n} (-1)^k \binom{4n+1}{2k+1}$$

Note that

$$\sum_{k=0}^{2n} (-1)^k \binom{4n+1}{2k+1}$$

is the sum we want to find, it remains to find the imaginary part of

$$(1+i)^{4n+1}$$

Recall

$$(1+i)^{4n} = (2i)^{2n} = (-2^2)^n = (-1)^n \times 2^{2n}$$

Then

$$(1+i)^{4n+1} = (-1)^n \times 2^{2n} + i \times (-1)^n \times 2^{2n}$$

Hence, the answer is

$$(-1)^n \times 2^{2n}$$

2 Intermediate level questions

Q5a) Show that,for any natural number n,r with $n \geq r$, we have

$$r \times \binom{n}{r} = n \times \binom{n-1}{r-1}$$

$$r \times \binom{n}{r} = \frac{n!}{r! \times (n-r)!} \times r = \frac{n!}{(r-1)! \times (n-r)!} = n \times \frac{(n-1)!}{(r-1)! \times [(n-1)-(r-1)]!} = n \times \binom{n-1}{r-1}$$

Q5b) By part a), or otherwise, show that

$$\sum_{r=0}^{n} r \times \binom{n}{r} = n \times 2^{n-1}$$

$$\sum_{r=0}^{n} r \times \binom{n}{r} = 0 + \sum_{r=1}^{n} n \times \binom{n-1}{r-1} = \sum_{r=1}^{n} n \times \binom{n-1}{r-1}$$

Note that,

$$\sum_{r=1}^{n} \binom{n-1}{r-1} = \sum_{r=0}^{n-1} \binom{n-1}{r} = 2^{n-1}$$

Hence,

$$\sum_{r=0}^{n} r \times \binom{n}{r} = \sum_{r=1}^{n} n \times \binom{n-1}{r-1} = n \times 2^{n-1}$$

Q5c) Can you also evaluate the sum

$$\sum_{r=0}^{n} r \times (r-1) \times \binom{n}{r}$$

By part a,

$$r \times (r-1) \times \binom{n}{r} = (r-1) \times n \times \binom{n-1}{r-1} = n \times (n-1) \times \binom{n-2}{r-2}$$

Hence,

$$\sum_{r=0}^{n} r \times (r-1) \times \binom{n}{r} = \sum_{r=0}^{n} n \times (n-1) \times \binom{n-2}{r-2} = \sum_{r=2}^{n} n \times (n-1) \times \binom{n-2}{r-2} = n \times (n-1) \times 2^{n-2}$$

Q6a) Show that,for any natural number n,r with $n \geq r$, we have

$$\binom{n}{r} \times \frac{1}{r+1} = \binom{n+1}{r+1} \times \frac{1}{n+1}$$

$$\binom{n}{r} \times \frac{1}{r+1} = \frac{n!}{r! \times (n-r)! \times (r+1)} = \frac{n!}{(r+1)! \times (n-r)!} = \frac{(n+1)!}{(r+1)! \times [(n+1)-(r+1)]!} \times \frac{1}{n+1}$$
$$= \binom{n+1}{r+1} \times \frac{1}{n+1}$$

Q6b) Hence, or otherwise, show that

$$\sum_{k=0}^{n} \binom{n}{r} \times \frac{1}{r+1} = \frac{2^{n+1}}{n+1}$$

By part (a),

$$\sum_{k=0}^{n} \binom{n}{r} \times \frac{1}{r+1} = \sum_{k=0}^{n} \binom{n+1}{r+1} \times \frac{1}{n+1} = \frac{1}{n+1} \times \sum_{k=0}^{n} \binom{n+1}{r+1} = \frac{1}{n+1} \times 2^{n+1}$$

Q7) By considering $(1+x)^{2n} = (1+x)^n \times (1+x)^n$, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2}$$

We consider the coefficient of x^n for both sides, for left hand side, the coefficient is

$$\binom{2n}{n}$$

, which is

$$\frac{(2n)!}{n! \times n!}$$

For right hand side, the x^n terms are formed by the multiplication of x^0 with x^n,x^1 with x^{n-1},x^12 with x^{n-2} and so on. Therefore, the coefficient of x^n of right hand side is

$$\binom{n}{0}\times\binom{n}{n}+\binom{n}{1}\times\binom{n}{n-1}+\binom{n}{2}\times\binom{n}{n-1}+...+\binom{n}{n-1}\times\binom{n}{1}+\binom{n}{n}\times\binom{n}{0}$$

Recall that

$$\binom{n}{n}=\binom{n}{0}$$

,

$$\binom{n}{n-1}=\binom{n}{1}$$

and so on. (More rigorously,we will say $\binom{n}{k} = \binom{n}{n-k}$ for any natural number $k = 0, 1, 2, ..., n$) Therefore,

$$\binom{n}{0}\times\binom{n}{n}+\binom{n}{1}\times\binom{n}{n-1}+...+\binom{n}{n-1}\times\binom{n}{1}+\binom{n}{n}\times\binom{n}{0}$$

$$=\binom{n}{0}\times\binom{n}{0}+\binom{n}{1}\times\binom{n}{1}+...+\binom{n}{n}\times\binom{n}{n}=\binom{n}{0}^2+\binom{n}{1}^2+...+\binom{n}{n-1}^2+\binom{n}{n}^2$$

Hence,

$$\binom{n}{0}^2+\binom{n}{1}^2+...+\binom{n}{n-1}^2+\binom{n}{n}^2=\frac{(2n)!}{(n!)^2}$$

Q8) By considering $(1+x)^{2n} = (1+x)^n \times (1+x)^n$ and the coefficient of x^{n+2},show that

$$\sum_{k=0}^{n-2}\binom{n}{k}\times\binom{n}{k+2}=\frac{(2n)!}{(n-2)!\times(n+2)!}$$

For LHS, the coefficient of x^{n-2} is

$$\binom{2n}{n-2}=\frac{(2n)!}{(n-2)!\times[2n-(n-2)]!}$$

. For RHS, the coefficient of x^{n-2} is

$$\binom{n}{0}\times\binom{n}{n-2}+\binom{n}{1}\times\binom{n}{n-3}+...+\binom{n}{n-2}\times\binom{n}{0}=\binom{n}{0}\times\binom{n}{2}+\binom{n}{1}\times\binom{n}{3}+...+\binom{n}{n-2}\times\binom{n}{n}$$

$$= \sum_{k=0}^{n-2} \binom{n}{k} \times \binom{n}{k+2}$$

Remark: Note that $\binom{n}{k+2} = \binom{n}{n-k-2}$ and $k+(n-k-2) = n-2$, this suggests we may consider the coefficient of x^{n-2}

Q9) In the lesson, we have learnt to find the sum

$$\sum_{r=0}^{n} \binom{2n}{2r} \times = 2^{(2n-1)}$$

. We now learn another way to evaluate the sum.

Q9a) Expand$(1+x^2)^{2n}$, by Binomial theorem.

$$(1+x^2)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \times x^{2k}$$

Q9b) by putting $x = 1$ and $x = i$ in the expansion in (a), show that

$$\sum_{r=0}^{2n} \binom{2n}{r} = 2^{2n}$$

and

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} = 0$$

Put $x = 1$

$$(1+1^2)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} \times 1^{2k}$$

$$2^{2n} = \sum_{r=0}^{2n} \binom{2n}{r}$$

Put $x = i$

$$(1+i^2)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} \times i^{2r}$$

$$0 = (1-1)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} \times (-1)^r$$

Q9c) By using part b), evaluate the sum

$$\sum_{r=0}^{n} \binom{2n}{2r}$$

and express your answer in terms of n
Taking sum of the two equations in part b, we get

$$2 \times \sum_{r=0}^{n} \binom{2n}{2r} = 2^{2n}$$

Hence,

$$\sum_{r=0}^{n} \binom{2n}{2r} = 2^{2n-1}$$

3 Challenging questions

Q10a) Let, m and n be positive integers, express the sum$\sum_{r=0}^{m} (1+x)^{n+r}$ in the form of

$$\frac{(1+x)^{A+B+1} - (1+x)^{C}}{x}$$

where A,B,C are some numbers (may express it in terms of m,n)

It is the sum of GS, so we put

$$\sum_{r=0}^{m} (1+x)^{n+r} = \frac{(1+x)^{n+r+1} - (1+x)^{n}}{(1+x)-1} = \frac{(1+x)^{n+m+1} - (1+x)^{n}}{x}$$

Q10b) By part a), show that

$$\binom{n}{n}+\binom{n+1}{n}+\binom{n+2}{n}+\binom{n+3}{n}+...+\binom{n+m-1}{n}+\binom{n+m}{n} = \binom{n+m+1}{n+1}$$

We consider the coefficient of x^n for both sides.
For LHS, the coefficient of x^n is the coefficient of x^n of each term. that is:

$$\binom{n}{n} + \binom{n+1}{n} + ... + \binom{n+m}{n}$$

For RHS, the coefficient of $x^n$is the coefficient of x^{n+1} of numerator. that is

$$\binom{n+m+1}{n+1}$$

Q10c) By part b), show that

$$\sum_{k=4}^{n+3} k(k-1)(k-2) = 6 \times (\binom{n+4}{4} - 1)$$

Hence evaluate the sum

$$\sum_{k=0}^{m} k(k-1)(k-2)$$

for $m \geq 3$

put $m = 3$ in part (b), we get

$$\sum_{k=3}^{n+3} \binom{k}{3} = \binom{n+4}{4}$$

$$\binom{3}{3} + \sum_{k=4}^{n+3} \binom{k}{3} = \binom{n+4}{4}$$

$$\sum_{k=4}^{n+3} k \times (k-1) \times (k-2) \div 6 = \binom{n+4}{4} - 1$$

$$[\sum_{k=4}^{n+3} k \times (k-1) \times (k-2)] \div 6 = \binom{n+4}{4} - 1$$

$$\sum_{k=4}^{n+3} k \times (k-1) \times (k-2) = 6 \times [\binom{n+4}{4} - 1]$$

Therefore, if we put $m = n + 3$, we get

$$\sum_{k=4}^{m} k \times (k-1) \times (k-2) = 6 \times [\binom{m+1}{4} - 1]$$

$$\sum_{k=0}^{m} k(k-1)(k-2) = 0 + 0 + 0 + 3 \times 2 \times 1 + \sum_{k=4}^{n+3} k \times (k-1) \times (k-2)$$

$$= 6 + 6 \times (\binom{m+1}{4} - 1) = 6 \times \binom{m+1}{4}$$

Q10d) Can you also evaluate the sum

$$\sum_{k=0}^{m} k(k-1)(k-2)(k-3)$$

and

$$\sum_{k=0}^{m} k(k-1)(k-2)(k-3)(k-4)$$

for $m \geq 4$ and $m \geq 5$ respectively?
(Question modified from HKAL 1994)

For the first expression,put $m = 4$ and repeat the steps as in part c, we can see that

$$\sum_{k=0}^{m} k(k-1)(k-2)(k-3) = 24 \times \binom{m+1}{5}$$

Similarly, put $m = 5$,we get

$$\sum_{k=0}^{m} k(k-1)(k-2)(k-3) = 120 \times \binom{m+1}{6}$$

Q11a) Find all the roots of the equation $1 + x + x^2 + x^3 = 0$

$$1 + x + x^2 + x^3 = (1 + x) + x^2 \times (1 + x) = (1 + x)(1 + x^2)$$

Hence

$$(1 + x)(1 + x^2) = 0$$

$x = -1$ or $x^2 = -1$ $x = -1$ or $x = i$ or $x = -i$
Q11b) By using part a and considering $(1 + x + x^2 + x^3)^{4n}$, or otherwise, find the sum of the following expression(x_r represent the coefficient of x^r of $(1+x+x^2 + x^3)^{4n}$):
bi)

$$\sum_{r=0}^{4n} (-1)^r x_r$$

Write $(1 + x + x^2 + x^3)^{4n} = \sum_{r=0}^{4n} x_r \times x^r$ If we put the $x = -1$ into the above equation, we get

$$\sum_{r=0}^{4n} (-1)^r \times x_r = 0$$

bii)

$$\sum_{r=0}^{n} x_{4r}$$

If we put the $x = i$ into the equation, we get

$$\sum_{r=0}^{4n} i^r \times x_r = 0$$

$$\sum_{r=0}^{2n}(-1)^r \times x_{2r} + i \times \sum_{r=0}^{2n-1}(-1)^r \times x_{2r+1} = 0$$

By comparing the real part an the imaginary part, we get

$$\sum_{r=0}^{2n}(-1)^r \times x_{2r} = 0$$

and

$$\sum_{r=0}^{2n-1}(-1)^r \times x_{2r+1} = 0$$

Therefore the answer of part bii,biii,biv and bv are equal,put $x =$ in the expansion, we can see their sum are 4^{4n} therefore, the answer for these part are all $4^{4n} \div 4 = 4^{4n-1}$

Q12) By considering $(1 + x + x^2)^{3n}$, show that:

a)

$$\sum_{r=0}^{n} x_{3r} = 3^{3n-1}$$

b)

$$\sum_{r=0}^{n-1} x_{3r+1} = 3^{3n-1}$$

c)

$$\sum_{r=0}^{n-1} x_{3r+2} = 3^{3n-1}$$

where x_r represent the coefficient of x^r of $(1 + x + x^2)^{3n}$

solve $1 + x + x^2 = 0$,we get

$$x = \frac{-1 + \sqrt{3}i}{2}$$

,we call this ω note that both ω and ω^2 satisfy the equation $1 + x + x^2 = 0$ because $\omega^3 = 1$, then we can put 1,ω and ω^2 into the expansion

$$(1 + x + x^2)^{3n} = \sum_{r=0}^{3n} x_r \times x^r$$

, we get

$$3^{3n} = \sum_{r=0}^{3n} x_r = x_0 + x_1 + x_2 + x_3 + ... + x_{3n-1} + x_{3n} - - - - - (1)$$

$$0 = x_0 + x_1 \times \omega + x_2 \times \omega^2 + x_3 \times \omega^3 + ... + x_{3n} \times \omega^{3n}$$
$$= x_0 + x_1 \times \omega + x_2 \times \omega^2 + x_3 \times 1 + ... + x_{3n} \times 1 - - - - - (2)$$

$$0 = x_0 + x_1 \times \omega^2 + x_2 \times \omega^4 + x_3 \times \omega^6 + ... + x_{3n} \times \omega^{6n}$$
$$= x_0 + x_1 \times \omega^2 + x_2 \times \omega + x_3 \times 1 + ... + x_{3n} \times 1 - - - - - (3)$$

adding up (1),(2) and (3), we get

$$3^{3n} = 3 \times x_0 + (1 + \omega + \omega^2) \times x_1 + (1 + \omega^2 + \omega^1) \times x_2 + 3 \times x_3 + ... + 3 \times x_{3n}$$

Recall $1 + \omega + \omega^2 = 0$

$$3^{3n} = 3 \times x_0 + 3 \times x_3 + 3 \times x_6 + ... + 3 \times x_{3n}$$

$$3^{3n-1} = x_0 + x_3 + x_6 + ... + x_{3n}$$

adding up (1),(2) $\times \omega^2$ and (3) $\times \omega$ we get

$$3^{3n} = (1+\omega^2+\omega)\times x_0+(1+\omega^3+\omega^3)\times x_1+(1+\omega+\omega^2)\times x_2+(1+\omega^2+\omega)\times x_3+(1+\omega^3+\omega^3)\times x_4+...+(1+\omega^2+\omega)\times x_{3n}$$

$$3^{3n} = 3 \times x_1 + 3 \times x_4 + 3 \times x_7 + ... + 3 \times x_{3n-2}$$

$$3^{3n-1} = x_1 + x_4 + x_7 + ... + x_{3n-2}$$

Similarly, adding up (1),(2) $\times \omega$ and (3) $\times \omega^2$ we get

$$3^{3n-1} = x_2 + x_5 + x_8 + ... + x_{3n-1}$$