Solution 6

p. 183: 1, 2, 6

1. Find the closest point in the plane $2x + y - 3z = 0$ to a point $\mathbf{x} \in \mathbb{R}^3$. (Hint: Find $M^\perp$.)

   Solution. Let M be the subspace $2x + y - 3z = 0$, which has unit normal vector $\mathbf{n} = \frac{(2,1,-3)}{\sqrt{2^2+1^2+(-3)^2}} = \frac{1}{\sqrt{14}}(2,1,-3)$. Then $M = \{\mathbf{n}\}^\perp$.

   Let $\mathbf{x} = (x_0, y_0, z_0)$. Then the closest point in M to $\mathbf{x}$ is given by

$$\begin{aligned} P\mathbf{x} &= \mathbf{x} - \langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n} \\ &= (x_0, y_0, z_0) - \langle (x_0, y_0, z_0), \frac{1}{\sqrt{14}}(2,1,-3)\rangle \frac{1}{\sqrt{14}}(2,1,-3) \\ &= (x_0, y_0, z_0) - \frac{2x_0 + y_0 - 3z_0}{14}(2,1,-3) \\ &= \frac{1}{14}(10x_0 - 2y_0 + 6z_0, -2x_0 + 13y_0 + 3z_0, 6x_0 + 3y_0 + 5z_0) \end{aligned}$$

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2. Let (a) $M := [\![y]\!]$, or (b) $M := \{y\}^\perp$, where y is a unit vector. The orthogonal projection P which maps any point x to its closest point in M is (a) $Px = \langle y, x\rangle y$, (b) $Px = x - \langle y, x\rangle y$.

   Solution. (a) Since $Px \in M$, we have $Px = \lambda y$ for some scalar λ. Also $x - Px \in M^\perp$. Hence

$$0 = \langle y, x - Px\rangle = \langle y, x\rangle - \lambda\langle y, y\rangle = \langle y, x\rangle - \lambda.$$

   Thus $\lambda = \langle y, x\rangle$ and $Px = \langle y, x\rangle y$.

   (b) Note that $x - Px \in M^\perp = (\{y\}^\perp)^\perp = [\![y]\!]$, so that $x - Px = \lambda y$, for some scalar λ. As $Px \in M$, we have

$$0 = \langle y, x\rangle = \langle y, x - \lambda y\rangle = \lambda - \langle y, x\rangle$$

   Again $\lambda = \langle y, x\rangle$ and hence $Px = x - \langle y, x\rangle y$.

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6. Let T be a square matrix, and suppose both subspaces M and $M^\perp$ are T-invariant, so that T takes the schematic form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Show that $\|T\| = \max(\|A\|, \|B\|)$. (Hint: Take $x = a + b$, then $\|Tx\|^2 = \|Ta\|^2 + \|Tb\|^2$.)

Solution. Take $x = a + b$, where $a \in M$ and $b \in M^\perp$. Then $Tx = Ta + Tb$. Since M and $M^\perp$ are T-invariant, we have $Ta = Aa \in M$ and $Tb = Bb \in M^\perp$. Now, by Pythagoras' theorem,

$$\|Tx\|^2 = \|Ta\|^2 + \|Tb\|^2 \quad \text{and} \quad \|x\|^2 = \|a\|^2 + \|b\|^2.$$

Hence

$$\begin{aligned}\frac{\|Tx\|^2}{\|x\|^2} &= \frac{\|Ta\|^2 + \|Tb\|^2}{\|a\|^2 + \|b\|^2} \\ &\leq \frac{\|A\|^2\|a\|^2 + \|B\|^2\|b\|^2}{\|a\|^2 + \|b\|^2} \\ &= t\|A\|^2 + (1-t)\|B\|^2, \qquad \left(\text{ where } t = \frac{\|a\|^2}{\|a\|^2 + \|b\|^2} \in [0,1]\right) \\ &\leq \max(\|A\|^2, \|B\|^2).\end{aligned}$$

By taking supremum over $\|x\| = 1$, we have $\|T\| \leq \max(\|A\|, \|B\|)$.

On the other hand, for any $a \in M$, $b \in M^\perp$, we have

$$\|Aa\| \leq \|T\|\|a\| \quad \text{and} \quad \|Bb\| \leq \|T\|\|b\|,$$

so that $\|A\| \leq \|T\|$ and $\|B\| \leq \|T\|$. Hence $\|T\| = \max(\|A\|, \|B\|)$. ◀