THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 Stochastic Processes, 2023-23 Term 2

Test 1 Reference Solution

Note: In what follows, for a given Markov chain $\{X_n\}_{n\geq 0}$ with a state space S, we always assume that it is time-homogeneous in the sense that $P(x,y) = P(X_{n+1} = y|X_n = x)$ for any $n = 0, 1, 2, \cdots$ and for any states x and y in S.

1. *(15pts)* Point out TRUE or FALSE for each statement. No need to justify your answers.

   (a) The sum of two discrete Poisson random variables is also a discrete Poisson random variable.

   Ans: True. For any two Poisson random variables: $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$, the sum of those two random variables is another Poisson: $X + Y \sim Poi(\lambda_1 + \lambda_2)$.

   (b) It holds that

$$P(X_3 = y|X_1 = x) = \sum_{x_2 \in S} P(X_3 = y, X_2 = x_2|X_0 = x_0, X_1 = x),$$

   where x_0 is an arbitrary state in S.

   Ans: True. Since $\{X_n\}$ is a Markovian process, $P(X_3 = y|X_1 = x) = P(X_3 = y|X_1 = x, X_0 = x_0) = \sum_{x_2 \in S} P(X_3 = y, X_2 = x_2|X_0 = x_0, X_1 = x)$.

   (c) If a state $x \in S$ satisfies $P(X_n = x \text{ for some } n \geq 1|X_0 = x) > 0$, then x is recurrent.

   Ans: False.

   (d) For any integers $m \geq n \geq 0$, one has $P(X_m = y|X_n = x) = P^{m-n}(x, y)$.

   Ans: True.

   (e) Let $\rho_{xy} = P_x(T_y < \infty)$. If $\rho_{xx} > 0$ and $x \to y$ then $y \to x$ and $\rho_{yy} > 0$.

   Ans: False. We may assume a two state Markovian chain with P(0,0) = 0.5, P(0,1)=0.5, P(1,0)=0,P(1,1)=1 then if x=0, y=1, one has $\rho_{00} = 0.5$, and $0 \to 1$ but 1 does not lead to 0.

   (f) Let x be a transient state and $x \to y$. Then, y is also a transient state.

   Ans: False. Use the counterexample in (e), 0 is transient and $0 \to 1$ but 1 is an absorbing state.

(g) If $x \to y$ then $P(x,y) > 0$.

Ans: False. Use the Markovian process in Question 3 as a counterexample. $0 \to 1 \to 2$ but $P(0,2) = 0$.

(h) Let x, y be two states. If $x \to y$ and $y \to x$ then x and y must be in a closed set.

Ans: True.

(i) Let C be an irreducible closed set then all states in C must be recurrent.

Ans: False. A counterexample is the 3 dimensional Polya's walk from Tutorial 5.

(j) A Markov chain must contain at least one recurrent state.

Ans: False. A counterexample is the 3 dimensional Polya's walk from Tutorial 5.

2. *(20pts)* In each model, setup it as a Markov chain $\{X_n\}_{n\geq 0}$ with a state space S and find the transition matrix P:

(a) A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.

(b) Suppose a phone system can hold one call when it is busy. For each unit time there is a probability p that a call will come in (assume no two calls come in at the same time). If it is busy there is a probability q that the conversation will finish in the next unit time. Let $\{0, 1, 2\}$ be the three states: free, busy and no hold, busy and on hold.

Ans:

(a) The state space is $\{0, 1, 2\}$.

$$P = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \end{array}.$$

(b) For x = 0, we have

$$P(0,0) = 1 - p, \;\; P(0,1) = p, P(0,2) = 0.$$

For x=1 (busy and no hold):

$$P(1,0) = P(\text{finished and no new call}) = (1-p)q$$

$$P(1,2) = P(\text{not finished and receive a new call}) = p(1-q)$$

$$P(1,1) = 1 - P(1,0) - P(1,2) = pq + (1-p)(1-q)$$

For x=2 (busy and on hold):

$$P(2,0) = 0$$

$$P(2,1) = P(\text{finished and no new call}) = (1-p)q$$

$$P(2,2) = 1 - P(2,0) - P(2,1) = 1 + pq - q$$

then the transition matrix is

$$P = \begin{array}{c} \\ \\ \\ \end{array}\begin{bmatrix} \overset{0}{1-p} & \overset{1}{p} & \overset{2}{0} \\ (1-p)q & pq+(1-p)(1-q) & p(1-q) \\ 0 & (1-p)q & pq+1-q \end{bmatrix}.$$

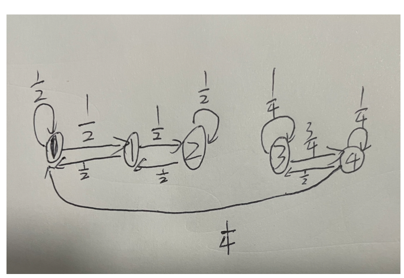
Added note: If we do not allow any new call to come in when there are already two calls on the line (one is busy and the other on hold), then we have in the last row: $P(2,0) = 0$, $P(2,1) = q$, and $P(2,2) = 1-q$.

3. *(15pts)* Consider the Markov chain on $S = \{0,1,2,3,4\}$ with the transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \end{array}.$$

Find the expected number of visits to state 3 starting from state 3.

Ans:



From the graph we can see that $0 \to 1 \to 2 \to 0$ and $0, 1, 2$ do not lead to any other state, so C1={0,1,2} is a closed irreducible set, and since it is finite, C1 is a closed irreducible set of recurrent state. State 3 and 4 might visit C1 and never come back, so $S_T = \{3, 4\}$ is a set of transient states.

Since state 3 is transient, we have $E_3[N(3)] = \frac{\rho_{33}}{1-\rho_{33}}$. By one step formula. we have

$$\begin{cases} \rho_{33} = P(3,3) + P(3,4)\rho_{43} \\ \rho_{43} = P(4,3) + P(4,4)\rho_{43} \end{cases}$$

hence $\rho_{33} = \frac{3}{4}$, then $E_3[N(3)] = 3$.

4. *(30pts)* Consider a Markov chain with state space $S = \{0, 1, 2, 3, 4, 5\}$ and transition matrix

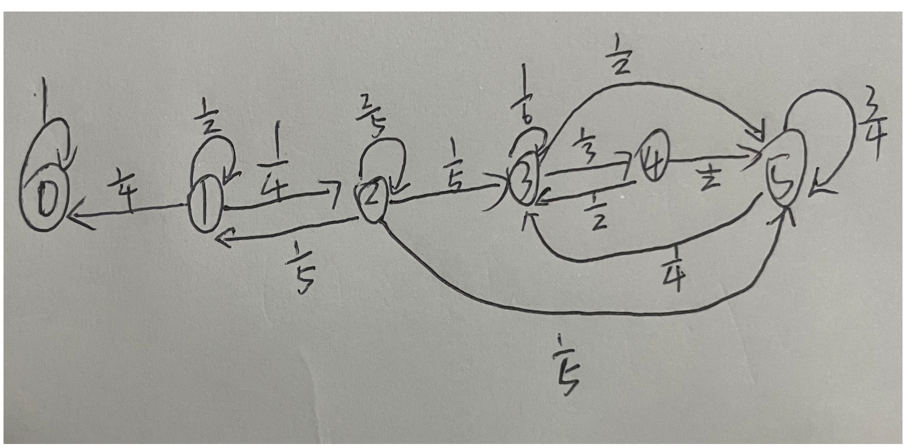
$$P = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 1/5 & 2/5 & 1/5 & 0 & 1/5 \\ 0 & 0 & 0 & 1/6 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/4 & 0 & 3/4 \end{bmatrix} \end{array}.$$

(a) Identify the transient and recurrent states, and the irreducible closed sets.

(b) Find $P(X_2 = 3 | X_0 = 3)$.

(c) What is the probability that the chain starting from state 1 eventually visits state 0?

Ans:



(a) From the graph we can see that state 0 is absorbing and hence recurrent, then C1={0} is a closed irreducible set of a single recurrent state. $3 \to 4 \to 5 \to 3$ and they do not lead to any other state, hence C2 = {3,4,5} is a closed irreducible set and since it is finite, C2 is a closed irreducible set of recurrent states. States 2 and 3 might visit C1 or C2 and never come back, so $S_T = \{1, 2\}$ is a set of transient states.

(b)

$$\begin{aligned}
&P(X_2 = 3|X_0 = 3) \\
&= \sum_{i \in S} P(X_2 = 3, X_1 = i|X_0 = 3) \\
&= \sum_{i \in S} P(3, i)P(i.3) \\
&= \frac{1}{6} * \frac{1}{6} + \frac{1}{3} * \frac{1}{2} + \frac{1}{2}\frac{1}{4} \\
&= \frac{23}{72}.
\end{aligned}$$

(c)

$$\begin{cases}
\rho_{10} = P(1,0) + P(1,1)\rho_{20} + P(1,2)\rho_{20}, \\
\rho_{20} = P(2,0) + P(2,1)\rho_{10} + P(2,2)\rho_{20}
\end{cases}$$

Solving above equation we have $\rho_{10} = \frac{3}{5}$

5. *(10pts)* Consider the birth and death chain on $\{0, 1, 2, \cdots\}$ defined by

$$p_x = \frac{x+2}{2(x+1)}, \quad q_x = \frac{x}{2(x+1)}, \quad x \geq 0,$$

where p_x is the probability of changing from x to $x+1$ and q_x is the probability of changing from x to $x - 1$. Determine whether the chain is recurrent or transient.

Ans:
Let $\gamma_0 = 1$, $\gamma_x = \frac{q_1 q_2 \cdots q_x}{p_1 p_2 \cdots p_x} = \frac{2}{(x+1)(x+2)} = 2(\frac{1}{x+1} - \frac{1}{x+2})$, since $\sum_{y=1}^{\infty} = \sum_{y=1}^{\infty} 2(\frac{1}{x+1} - \frac{1}{x+2}) = 1 < \infty$, so the chain is transient.

6. *(10pts)* Consider the aging chain on $\{0, 1, 2, ...\}$ in which for any $n \geq 0$ the individual gets 1 day older from n to $n + 1$ with probability p_n but dies and returns to age 0 with probability $1 - p_n$. Find a condition on the sequence $(p_n)_{n \geq 0}$ such that 0 is recurrent.

Ans:

Methods 1:
State 0 is recurrent $\iff \rho_{00} = 1$. Note that $\rho_{00} = P(0,0) + P(0,1)\rho_{10}$ and for any n,

$\rho_{n0} = P(n,0) + P(n,n+1)\rho_{N+1,0} = 1 - p_n + \rho_{N+1,0}$, we have

$$\begin{aligned}
\rho_{00} &= P(0,0) + P(0,1)\rho_{10} \\
&= P(0,0) + P(0,1)(1 - p_1 + P(1,2)\rho_{20}) \\
&= 1 - p_0 + p_0(1-p_1) + p_0p_1(1 - p_2 + P(2,3)\rho_{30}) \\
&= ...... \\
&= 1 - \Pi_{i=0}^{n} p_i + \Pi_{i=0}^{n} p_i \rho_{(n+1),0}
\end{aligned}$$

Hence, to make state 0 recurrent, a condition can be there exists a state n_0 such that $\Pi_{i=0}^{n_0} p_i = 0$. However, it is not the optimal condition cause if there exists a n_0 such that $p_{n_0} = 0$, it becomes a trivial case.
Since $0 \leq \Pi_{i=0}^{n} p_i - \Pi_{i=0}^{n} p_i \rho_{(n+1),0} = \Pi_{i=0}^{n} p_i(1 - \rho_{(n+1),0}) \leq \Pi_{i=0}^{n} p_i$, if $\Pi_{i=0}^{\infty} p_i = 0$, then $\rho_{00} = 1$ and state 0 is recurrent.

Method 2 :

We can see that when $n = 1, P_0(T_0 = 1) = 1 - p_0$ and when $n > 1$,

$$P_0(T_0 = n) = P(0,1)P(1,2)\cdots P(n-2,n-1)P(n-1,0) = p_0p_1\cdots p_{n-2}(1 - p_{n-1}).$$

By definition,

$$\rho_{00} = P_0(T_0 < \infty) = \sum_{n=1}^{\infty} P_0(T_0 = n) = \lim_{k\to\infty} \sum_{n=1}^{k} P_0(T_0 = n).$$

Because

$$\begin{aligned}
\sum_{n=1}^{k} P_0(T_0 = n) &= 1 - p_0 + \sum_{n=2}^{k} p_0p_1\cdots p_{n-2}(1 - p_{n-1}) \\
&= 1 - p_0 + \sum_{n=2}^{k} p_0p_1\cdots p_{n-2} - \sum_{n=2}^{k} p_0p_1\cdots p_{n-2}p_{n-1} \\
&= 1 - p_0 + \sum_{n=1}^{k-1} p_0p_1\cdots p_{n-1} - \sum_{n=2}^{k} p_0p_1\cdots p_{n-2}p_{n-1} \\
&= 1 - p_0 + p_0 - p_0p_1\cdots p_{k-1} \\
&= 1 - p_0p_1\cdots p_{k-1},
\end{aligned}$$

we have

$$\rho_{00} = 1 - \lim_{k\to\infty} p_0p_1\cdots p_{k-1} = 1 - \lim_{k\to\infty} p_0p_1\cdots p_k.$$

Therefore, 0 is recurrent if and only if the infinite product $\Pi_{k=0}^{\infty} p_k$ converges to 0.

—THE END—