THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2022/23 Term 2

Homework 1
Due date: January 20, 2023

Please hand in your answers on Blackboard to all questions below.

Q1. (a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_x^1 6(1-y)\, dy = 3(1-x)^2,\ x \in [0,1]$.

(b) $f_x(x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{6(1-y)}{6(y-y^2)} = \frac{1}{y}\ for\ 0 \le x \le y \le 1$.

(c) Since $f_X(x) \cdot f_Y(y) \neq f_{X,Y}(x,y)$, X, Y are not independent.

(d) $f_Y(y|X=\frac{1}{2}) = 8(1-y)\ for\ y \in [x,1]$,
then $P(Y \ge \frac{3}{4}|X=\frac{1}{2}) = \int_{3/4}^1 8(1-y)\, dy = 1/4$.

(e) $E[X-3Y] = E[X] - 3E[Y] = -\frac{5}{4}$.

Q2. (a) Since X, Y are independent, one has

$$P(Z \le \frac{1}{2}, X=0) = P(Z \le \frac{1}{2})P(X=0) = 1/6.$$

Then $P(Z \le \frac{1}{2}|X=0) = P(Z \le \frac{1}{2}, X=0)/P(X=0) = 1/2$.

(b) For $z \in [-1,2]$, one has

$$\begin{aligned}
&P(Z \le z)\\
&= P(X+Y \le z)\\
&= P(X+Y \le z|X=-1)P(X=-1) + P(X+Y \le z|X=0)P(X=-1)\\
&\quad + P(X+Y \le z|X=1)P(X=1)\\
&= \frac{1}{3}P(Y \le z+1) + \frac{1}{3}P(Y \le z) + \frac{1}{3}P(Y \le z-1)\\
&= \begin{cases} \frac{1}{3}\int_0^{z+1} ds + 0 + 0 & \text{if } -1 \le z < 0,\\ \frac{1}{3} + \frac{1}{3}\int_0^{z} ds + 0 & \text{if } 0 \le z < 1,\\ \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\int_0^{z-1} ds & \text{if } 1 \le z \le 2, \end{cases}\\
&= \frac{1+z}{3}
\end{aligned}$$

Hence, pdf of Z equals to 1/3 when $Z \in [-1,2]$, otherwise, $f_Z(z) = 0$.