MATH4240: Stochastic Processes Tutorial 3

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06 February, 2023

Recurrent/ Transient

Denote by

$$\rho_{xy} = P_x(T_y < \infty) \overset{def}{=} P(T_y < \infty | X_0 = x)$$

the probability that the chain from x returns back to y in finite time.
If $\rho_{xx} = 1$, we call x a *recurrent* state. Otherwise, we call x a *transient* state.

Two-state Markov chain

Let $\{X_n\}_{n\geq 0}$ be the two-state Markov chain (page 2 in textbook) with the state space $\mathcal{S} = \{0, 1\}$ and the transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

where $0 < p, q < 1$.
Find ρ_{00}.

Two-state Markov chain

$$
\begin{aligned}
\rho_{00} &= P_0(T_0 < \infty) \\
&= P(X_1 = 0 | X_0 = 0) + P(X_1 = 1, X_2 = 0 | X_0 = 0) + \\
&\quad P(X_1 = X_2 = 1, X_3 = 0 | X_0 = 0) + \dots \\
&= (1-p) + pq + p(1-q)q + p(1-q)^2 q + \dots \\
&= (1-p) + pq \sum_{k=0}^{\infty} (1-q)^k \\
&= (1-p) + \frac{pq}{1-(1-q)} \\
&= 1
\end{aligned}
$$

Similarly. we also have $\rho_{11} = 1$, i.e. every state is recurrent.

One-Step Calculations on hitting probabilities

In the textbook, question 9 on page 42 says: using the formula

$$P_x(T_y = n+1) = \sum_{z \neq y} P(x,z) P_z(T_y = n), \quad n \geq 1 \tag{1}$$

to verify the following identity

$$\rho_{xy} = P(x,y) + \sum_{z \neq y} P(x,z) \rho_{zy}. \tag{2}$$

One-Step Calculations on hitting probabilities

Proof. By formula (2),

$$
\begin{aligned}
P_x(T_y < n+1) &= \sum_{k=0}^{n-1} P_x(T_y = k+1) \\
&= P_x(T_y = 1) + \sum_{k=1}^{n-1} P_x(T_y = k+1) \\
&= P(x,y) + \sum_{k=1}^{n-1} \left(\sum_{z \neq y} P(x,z) P_z(T_y = k) \right) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) \sum_{k=1}^{n-1} P_z(T_y = k) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) P_z(T_y < n), \quad n \geq 1.
\end{aligned}
$$

One-Step Calculations on hitting probabilities

As in the definition $\rho_{xy} = P_x(T_y < \infty)$, we have

$$
\begin{aligned}
\rho_{xy} &= \lim_{n\to\infty} P_x(T_y < n+1) \\
&= \lim_{n\to\infty} \left(P(x,y) + \sum_{z<y} P(x,z) P_z(T_y < n) \right) \\
&\qquad \text{(by Monotone Convergence Theorem)} \\
&= P(x,y) + \sum_{z\neq y} P(x,z) \left(\lim_{n\to\infty} P_z(T_y < n) \right) \\
&= P(x,y) + \sum_{z\neq y} P(x,z) P_z(T_y < \infty) \\
&= P(x,y) + \sum_{z\neq y} P(x,z) \rho_{zy}.
\end{aligned}
$$

One-Step Calculations on expected values of hitting times

In a *finite* irreducible Markov chain (in particular $\rho_{xy} = 1$ for all $x, y \in \mathcal{S}$), another important formula can be induced from the formula (2):

$$E_x(T_y) = 1 + \sum_{z \neq y} P(x, z) E_z(T_y). \tag{3}$$

Proof. In a finite irreducible MC, $\rho_{xy} = 1$ for any $x, y \in \mathcal{S}$. By formula (2),

$$\begin{aligned} E_x(T_y) &= \sum_{n=0}^{\infty}(n+1)P_x(T_y = n+1) \\ &= P_x(T_y = 1) + \sum_{n=1}^{\infty}(n+1)P_x(T_y = n+1) \\ &= P(x,y) + \sum_{n=1}^{\infty}(n+1)\left(\sum_{z\neq y} P(x,z)P_z(T_y = n)\right) \\ &\quad \text{(since the state space is finite)} \\ &= P(x,y) + \sum_{z\neq y} P(x,z)\left(\sum_{n=1}^{\infty}(n+1)P_z(T_y = n)\right) \end{aligned}$$

$$
\begin{aligned}
&= P(x,y) + \sum_{z\neq y} P(x,z)\left(\sum_{n=1}^{\infty} nP_z(T_y = n) + \sum_{n=1}^{\infty} P_z(T_y = n)\right) \\
&= P(x,y) + \sum_{z\neq y} P(x,z)\left(E_z(T_y) + P_z(T_y < \infty)\right) \\
&= P(x,y) + \sum_{z\neq y} P(x,z)\left(E_z(T_y) + \rho_{zy}\right) \\
&= P(x,y) + \sum_{z\neq y} P(x,z)\left(E_z(T_y) + 1\right) \\
&= \sum_{z\in\mathcal{S}} P(x,z) + \sum_{z\neq y} P(x,z)E_z(T_y) \\
&= 1 + \sum_{z\neq y} P(x,z)E_z(T_y).
\end{aligned}
$$

Example: Ehrenfest Chain

Consider the Ehrenfest chain with $d = 3$.

(a) Find $P_x(T_0 = n)$ for $x \in \{0, 1, 2, 3\}$ and $1 \leq n \leq 3$.

(b) Find ρ_{10}, ρ_{20}, and ρ_{30}.

(c) Find $E_3(T_0)$.

Ehrenfest Chain

Example: Ehrenfest Chain

Solution.

(a) The transition matrix is

	0	1	2	3
0	0	1	0	0
1	1/3	0	2/3	0
2	0	2/3	0	1/3
3	0	0	1	0

For $n = 1$, $P_x(T_0 = 1) = P(x, 0) = \begin{cases} 1/3, & \text{if } x = 1; \\ 0, & \text{otherwise.} \end{cases}$

Example: Ehrenfest Chain

For $n = 2$, by formula (1),

$$\begin{aligned} P_x(T_0 = 2) &= \sum_{y \neq 0} P(x,y) P_y(T_0 = 1) \\ &= \sum_{y \neq 0} P(x,y) P(y,0) \\ &= P(x,1)P(1,0) = \begin{cases} 1/3, & \text{if } x = 0; \\ 2/9, & \text{if } x = 2; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example: Ehrenfest Chain

For $n = 3$, by formula (1),

$$\begin{aligned} P_x(T_0 = 3) &= \sum_{y \neq 0} P(x,y) P_y(T_0 = 2) \\ &= P(x,2) P_2(T_0 = 2) = \begin{cases} 4/27, & \text{if } x = 1; \\ 2/9, & \text{if } x = 3; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example: Ehrenfest Chain

(b) By formula (2),

$$\begin{cases} \rho_{30} = P(3,0) + \sum_{z\neq 0} P(3,z)\rho_{z0} = \rho_{20}, \\ \rho_{20} = P(2,0) + \sum_{z\neq 0} P(2,z)\rho_{z0} = (2/3)\rho_{10} + (1/3)\rho_{30}, \\ \rho_{10} = P(1,0) + \sum_{z\neq 0} P(1,z)\rho_{z0} = 1/3 + (2/3)\rho_{20}. \end{cases}$$

Hence $\rho_{10} = \rho_{20} = \rho_{30} = 1$.

Example: Ehrenfest Chain

(c) By formula (3),

$$\begin{cases} E_3(T_0) = 1 + \sum\limits_{z \neq 0} P(3, z) E_z(T_0) = 1 + E_2(T_0), \\ E_2(T_0) = 1 + \sum\limits_{z \neq 0} P(2, z) E_z(T_0) = 1 + (2/3) E_1(T_0) + (1/3) E_3(T_0), \\ E_1(T_0) = 1 + \sum\limits_{z \neq 0} P(1, z) E_z(T_0) = 1 + (2/3) E_2(T_0). \end{cases}$$

Hence $E_3(T_0) = 10$.

Example: Duration of Fair Games.

Gamble's ruin problem Consider a gambler who starts with an initial fortune and then on each successive gamble either wins 1 or loses 1 with probabilities p and $1 - p$ respectively. Let X_n denote the total fortune after the nth gamble. The gambler's objective is to reach a total fortune of N, without first getting ruined (running out of money). The gambler stops playing after winning or getting ruined, whichever happens first.

Example: Duration of Fair Games.

Consider the gambler's ruin chain in which $P(i, i+1) = P(i, i-1) = 1/2$ for $0 < i < N$ and the end points are absorbing states: $P(0,0) = P(N,N) = 1$. Let $T = \min\{T_0, T_N\}$ be the time at which the chain enters an absorbing state. Find $E_n(T)$ for each $n \in \{1, 2, \cdots, N-1\}$.

Example: Duration of Fair Games.

Solution. By formula (3),

$$E_j(T) = \frac{1}{2}E_{j-1}(T) + \frac{1}{2}E_{j+1}(T) + 1, \quad j = 1, \cdots, N-1. \tag{4}$$

Let $h(j) = E_j(T) - E_{j-1}(T)$, $j = 1, \cdots, N$, then by (4),

$$h(j) = h(j+1) + 2.$$

Hence

$$0 = E_N(T) - E_0(T) = \sum_{j=1}^{N} h(j) = \sum_{j=1}^{N} (h(1) - 2(j-1)) = N \cdot h(1) - N(N-1)$$

which implies $h(1) = N - 1$. Therefore, for $n \in \{1, 2, \cdots, N-1\}$,

$$E_n(T) = \sum_{j=1}^{n} h(j) + E_0(T) = \sum_{j=1}^{n} (h(1) - 2(j-1)) = n(N-n).$$