MATH4240: Stochastic Processes Tutorial 1

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Probability Space

A probability space $(\Omega, \mathcal{F}, P)$ consists of three parts:

- A sample space Ω, which is the set of all possible outcomes.
- A set of events $\mathcal{F}$, where each event is a set containing zero or more outcomes, i.e., a subset of the sample space.
- The assignment of probabilities to the events, that is, a function $P : \mathcal{F} \to [0, 1]$.

Probability Space

A *random variable* on $(\Omega, \mathcal{F}, P)$ is a measurable function $X : \Omega \to A \subseteq \mathbb{R}$. Generally, we have two types based on the different choices of A.
Discrete type: $A = \mathbb{Z}, \mathbb{N}, \mathbb{N}_+, \ldots$
Continuous type: $A = \mathbb{R}, \mathbb{R}_+, [a, b], \ldots$

The following notion provides a way to relate an event to a random variable: the event

$$(X \in B) := \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

Probability Space

Probability mass function: $f_X(x) = P(X = x)$

Probability density function: $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$.

Cumulative distribution function: $F_X(x) = P(X \leq x)$.

Expected value $\mu = E[X]$ and **variance** $\sigma^2 = \text{Var}(X)$: for discrete random variables

$$\mu = \sum_i x_i\, P(X = x_i), \quad \sigma^2 = E[(X - \mu)^2] = \sum_i (x_i - \mu)^2\, P(X = x_i),$$

for continuous random variables

$$\mu = \int_{\mathbb{R}} x f_X(x)\,dx, \quad \sigma^2 = \int_{\mathbb{R}} (x - \mu)^2 f_X(x)\,dx.$$

Probability Space

Some notations The **Kronecker delta** is a function of two variables, usually non-negative integers, defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

The **indicator function** or a **characteristic function** of a subset A of a set X is a function $\mathbf{1}_A : X \to \{0, 1\}$ defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Examples of random variables

- **Discrete type**

  1. **Binomial random variables:** $X \sim B(n, p)$,

  $$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots, n.$$

  When $n = 1$, the binomial distribution is called a **Bernoulli distribution**.
  $E[X] = np, \mathrm{Var}(X) = np(1-p)$.

Examples of random variables

2. **Poisson random variables:** $X \sim \text{Poi}(\lambda)$,

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \cdots$$

λ, called the **rate parameter**, is the average number of events per time interval.
$E[X] = \text{Var}(X) = \lambda$.

Examples of random variables

3. **Geometric random variables:** $X \sim G(p)$,

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \cdots$$

$E[X] = \dfrac{1}{p}, \mathrm{Var}(X) = \dfrac{1 - p}{p^2}$.

Examples of random variables

- **Continuous type**

  1. **Uniform random variables:** $X \sim U(a,b)$,

$$f_X(x) = \frac{1}{b-a}\mathbf{1}_{[a,b]}.$$

$$E[X] = \frac{1}{2}(b+a), \quad \mathrm{Var}(X) = \frac{1}{12}(b-a)^2.$$

Examples of random variables

2. **Exponential random variables:** $X \sim \text{Exp}(\lambda)$,

$$f_X(x) = \lambda e^{-\lambda x} \mathbf{1}_{[0,\infty)}.$$

Events arrive at a rate λ (often called the **rate parameter**), when the time between events has a mean of $\frac{1}{\lambda}$.

$E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.$

3. **Normal random variables (Gauss distribution):** $X \sim N(\mu, \sigma^2)$,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

$E[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2.$

Functions of random variables

A new random variable Y can be defined by applying a Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ to the outcomes of random variables $X_1, X_2, \cdots, X_n$. That is, $Y = g(X_1, X_2, \cdots, X_n)$. The **cumulative distribution function** of Y is then

$$F_Y(y) = P(g(X_1, X_2, \cdots, X_n) \leq y).$$

As a special case, we will discuss the sum of two **independent** random variables.

Discrete type

Suppose that X and Y are two independent integer-valued random variables with probability mass function $f_X(x)$ and $f_Y(y)$ respectively. Let $Z = X + Y$ and we would like to determine the mass function $f_Z(z)$ of Z. Notice that the event $(Z = z)$ is the disjoint union of the events $(X = k) \cap (Z = z)$, where k runs over all integers.
Hence by independence of X and Y,

$$\begin{aligned} f_Z(z) = P(Z = z) &= \sum_{k\in\mathbb{Z}} P(Z = z, X = k) = \sum_{k\in\mathbb{Z}} P(X = k, Y = z - k) \\ &= \sum_{k\in\mathbb{Z}} P(X = k)P(Y = z - k) \\ &= \sum_{k\in\mathbb{Z}} f_X(k)f_Y(z - k) = (f_X * f_Y)(z), \quad z \in \mathbb{Z}. \end{aligned}$$

Here $f_Z = f_X * f_Y$ is called the (discrete) *convolution* of f_X and f_Y.

Exercise. If $X \sim \text{Poi}(\lambda)$, $Y \sim \text{Poi}(\mu)$ are independent, then $X + Y \sim \text{Poi}(\lambda + \mu)$.
By induction, the conclusion can be generalized as

$$X_i \sim \text{Poi}(\lambda_i), i = 1, 2, \cdots, m \text{ are independent}$$

implies that

$$Y = \sum_{i=1}^{m} X_i \sim \text{Poi}(\lambda_1 + \cdots + \lambda_m).$$

Continuous type

Suppose X and Y be two **independent** random variables with p.d.f. $f_X(x)$ and $f_Y(y)$. Let $Z = X + Y$. We want to find the p.d.f. $f_Z(z)$ of Z. By the independence of X and Y, their joint density function is $f_X(x)f_Y(y)$. Now compute $P(X + Y \leq z)$ by integrating joint density function,

$$P(X + Y \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x)f_Y(y)dxdy.$$

Differentiate with respect to variable z,

$$f_Z(z) = \frac{d}{dz}P(X + Y \leq z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy = (f_X * f_Y)(z).$$

Here $f_Z = f_X * f_Y$ is called the *convolution* of f_X and f_Y.

Functions of random variables

Let X_1, X_2, X_3, ... be a sequence of independent r.v.'s with the same p.d.f. $f(x)$ (that is, independent identically distribution random variables $=$ i.i.d.r.v.'s).
For $n \geq 1$, consider the r.v. $S_n = X_1 + X_2 + \cdots + X_n$, then $S_n = S_{n-1} + X_n$. By induction, the p.d.f of S_n is

$$f_{S_n} = f_{S_{n-1}} * f = f * f * \cdots * f \quad (n \text{ terms}).$$

This is the *n-fold convolution* of f.

Examples 1. Exponential random variables: Suppose $X_k \sim \text{Exp}(\lambda), k = 1, 2, 3, \cdots$ are independent, then the p.d.f. of $Z = X_1 + X_2$ is

$$f_Z(z) = \lambda^2 z e^{-\lambda z} \mathbf{1}_{[0,\infty)}.$$

Moreover, let $S_n = X_1 + X_2 + \cdots + X_n$, $n \geq 1$, then the p.d.f. of S_n is (verify it yourself)

$$f_{S_n}(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!}, & x \in [0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

2. More about exponential random variables. Suppose $X_i \sim \text{Exp}(\lambda_i), i = 1, 2, \cdots, n$ are independent, then we have the following important property, which $Y := \min(X_1, X_2, \cdots, X_n) \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n)$ is also an exponential random variable.

Proof: Recall that $f_{X_i}(x_i) = \lambda_i e^{-\lambda_i x_i} \mathbf{1}_{[0,\infty)}$, $F_{X_i}(x_i) = (1 - e^{-\lambda_i x_i})\mathbf{1}_{[0,\infty)}$ and then the distribution function of Y is $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= 1 - P(Y > y) \\ &= 1 - P(\min(X_1, X_2, \cdots, X_n) > y) \\ &= 1 - P(X_1 > y, X_2 > y, \cdots, X_n > y) \\ &= 1 - P(X_1 > y)P(X_2 > y) \cdots P(X_n > y) \\ &= 1 - (1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y)) \\ &= 1 - e^{-(\lambda_1 + \cdots + \lambda_n)y} \end{aligned}$$

and then the conclusion follows.

Functions of random variables

Another important property for exponential random variables are

$$P(X_k = \min(X_1, \cdots, X_n)) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}.$$

Proof: Let $Z = \min((X_1, \cdots, X_{k-1}, X_{k+1}, \cdots X_n))$. Then from the previous discussion we know $Z \sim \text{Exp}(\lambda)$ where $\lambda = \lambda_1 + \cdots + \lambda_{k-1} + \lambda_{k+1} + \cdots + \lambda_n$. Notice that X_k and Z are independent and consequently,

$$
\begin{aligned}
P(X_k = \min(X_1, \cdots, X_n)) = P(X_k \leq Z) &= \int_0^\infty \int_{x_k}^\infty f_{X_k}(x_k) f_Z(z)\, dz\, dx_k \\
&= \int_0^\infty \int_{x_k}^\infty \lambda_k e^{-\lambda_k x_k} \lambda e^{-\lambda z}\, dz\, dx_k \\
&= \int_0^\infty \lambda_k e^{-\lambda_k x_k} \cdot e^{-\lambda x_k}\, dx_k \\
&= \frac{\lambda_k}{\lambda_k + \lambda} = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}.
\end{aligned}
$$

An additional example. Let $X_k \sim \text{Exp}(\lambda_k), k = 1, 2$ be independent, find the p.d.f.'s of $Z = \max(X_1, X_2)$.
Solution: The cumulative density function of Z is

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(\max(X_1, X_2) \le z) = P(X_1 \le z, X_2 \le z) \\ &= \int_0^z \int_0^z \lambda_1 e^{-\lambda_1 x_1} \cdot \lambda_2 e^{-\lambda_2 x_2}\, dx_1\, dx_2 \\ &= (1 - e^{-\lambda_1 z})(1 - e^{-\lambda_2 z}), \qquad z > 0 \end{aligned}$$

and then

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \left[\lambda_1 e^{-\lambda_1 z} + \lambda_2 e^{-\lambda_2 z} - (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) z}\right] \mathbf{1}_{[0,\infty)}.$$

2. Sum of general random variables. Suppose X and Y are uncorrelated random variables (i.e. $E[XY] = E[X]E[Y]$), then

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y).$$

In particular, independent random variables satisfies the above assumption. Suppose further $X_i, i = 1, 2, \ldots, n$, are independent and have the same variance σ^2, denote by

$$\bar{X} := \frac{1}{n}\sum_{k=i}^{n} X_i$$

their mean. Then,

$$\text{Var}(\bar{X}) = \frac{1}{n}\sigma^2.$$