2. (a) The linear system can be rewritten as

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $L = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 2 & 2 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 2 & -2 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

Jacobi: $x^{k+1} = D^{-1}(-U-L)x^k + D^{-1}b$

Let $B_J = D^{-1}(-U-L) = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$

$C_J = D^{-1}b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$x^1 = B_J x^0 + C_J = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad x^2 = B_J x^1 + C_J = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

Gauss-Seidel: $x^{k+1} = (D+L)^{-1}(-U)x^k + (D+L)^{-1}b$

Let $B_{GS} = (D+L)^{-1}(-U) = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$

$C_{GS} = (D+L)^{-1}b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$x^1 = B_{GS}x^0 + C_{GS} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad x^2 = B_{GS}x^1 + C_{GS} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$

(b) Jacobi: $\det(B_J - \lambda I) = -\lambda^3$

~~eigen~~ $\Rightarrow \rho(B_J) = 0 \Rightarrow$ Jacobi method converges

Gauss-Seidel: $\det(B_{GS} - \lambda I) = -\lambda(\lambda-2)^2$

$\Rightarrow \rho(B_{GS}) = 2 \Rightarrow$ G.S. diverges

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3. (a) $B_{GS} = -(D+L)^{-1}U = -\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ -2 & -1 & 1 & \\ -3 & -2 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$= -\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & a+b \\ 0 & 0 & 0 & 3a+b+c \\ 0 & 0 & 0 & 8a+3b+c \end{pmatrix}$$

$\det(B_{GS} - \lambda I) = \lambda^3 (8a+3b+c+\lambda)$

Hence $\rho(B_{GS}) = |8a+3b+c|$.

GS converges $\Leftrightarrow$ $|8a+3b+c| < 1$

(b) $a = b = -c/11 \Rightarrow 8a+3b+c = 0$
$\Rightarrow \rho(B_{GS}) = 0$

Hence G-S converges in finitely many steps.

4 Counter example : $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

A is SPD and $\rho(B_J) \simeq 1.4051$

5. <u>Householder-John Theorem</u>:

If A and (N^T+N-A) are symmetric, positive definite real matrices then the iterative scheme

$$N x^{k+1} = p x^k + b \quad \text{converges.}$$

By Gershgorin Theorem, any eigenvalue of A, $\lambda \in [2, 6]$

Hence all eigenvalues of A are +ve, and A is SPD.

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Jacobi : $N^T + N - A = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$

is SPD (By Gershgorin Thm , all eigenvalues in $[2, 6]$)

Gauss - Seidel : $N^T + N - A = \begin{pmatrix} 4 & & & \\ & 4 & & \\ & & 4 & \\ & & & 4 \end{pmatrix}$ is SPD

(all eigenvalues are 4 , +ve).

$\therefore$ Both Jacobi and G-S converges.

6. (a) Let $D = \begin{pmatrix} 2 & & \\ & 3 & \\ & & 2 \end{pmatrix}$, $L = \begin{pmatrix} 0 & & \\ 2 & 0 & \\ 1 & 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

The SOR method is given by

$$x^{k+1} = \left(L + \tfrac{1}{\omega} D\right)^{-1} \left(\tfrac{1}{\omega} D - (D+U)\right)\left(x^k + b\right)$$

$$= \begin{pmatrix} 1-\omega , & -\omega/2 & \omega/2 \\ \frac{2\omega(\omega-1)}{3} & \frac{\omega^2}{3} - \omega + 1 & -\omega(\omega+1)/3 \\ \frac{-\omega(2\omega^2 - 5\omega + 3)}{6} & \frac{-\omega(2\omega^2 - 9\omega + 6)}{12} & \frac{\omega^3}{6} - \frac{\omega^2}{12} + 1 \end{pmatrix} x^k$$

$$+ \begin{pmatrix} 8/\omega - 11 \\ 24/\omega - 29 \\ 10/\omega - 10 \end{pmatrix}$$

(b) Let λ be an eigenvalue of

$$B = \left(L + \tfrac{1}{\omega} D\right)^{-1} \left(\tfrac{1}{\omega} D - (D+U)\right)$$

We have

$$\det\left(\left(L + \tfrac{1}{\omega} D\right)^{-1} \left(\tfrac{1}{\omega} D - (D+U)\right) - \lambda I\right) = 0$$

$$\det\left(\tfrac{1}{\omega} D - (D+U) - \lambda\left(L + \tfrac{1}{\omega} D\right)\right) = 0$$

Let $\tilde{B}_\lambda = \frac{1}{\omega} D - (D+U) - \lambda\left(L + \frac{1}{\omega} D\right)$

$$\tilde{B}_\lambda = \left(\frac{1}{\omega}(1-\lambda) - 1\right) D - U - \lambda L$$

Note that $\tilde{B}_\lambda$ has a zero eigenvalue. — $(*)$

Let η be an eigenvalue of $\tilde{B}_\lambda$. By Gershgorin Thm,

Case 1: $\lambda \le -1$

$$\eta \geq \left|\frac{1}{\omega}(1-\lambda) - 1\right| |a_{ii}| - \sum_{j<i} |a_{ij}| - |\lambda| \sum_{j>i} |a_{ij}|$$

$$= \left(\frac{1}{\omega} - 1\right) |a_{ii}| - \sum_{j<i} |a_{ij}| - \frac{\lambda}{\omega}\left(|a_{ii}| - \omega \sum_{j>i} |a_{ij}|\right)$$

$$\geq |a_{ii}| - \sum_{j<i} |a_{ij}| + \left(|a_{ii}| - \sum_{j>i} |a_{ij}|\right)$$

$$> 0 \qquad (A \text{ is SDD})$$

Case 2: $\lambda \ge 1$

$$\eta \le \left|\frac{1}{\omega}(1-\lambda) - 1\right| |a_{ii}| + \sum_{j<i} |a_{ij}| + |\lambda| \sum_{j<i} |a_{ij}|$$

$$= \left(\frac{1}{\omega}(1-\lambda) - 1\right) |a_{ii}| + \sum_{j<i} |a_{ij}| + \lambda \sum_{j<i} |a_{ij}|$$

$$\leq \left(\frac{1}{\omega}(1-\lambda) - 1\right) |a_{ii}| + \lambda \sum_{j<i} |a_{ij}| + \lambda \sum_{j<i} |a_{ij}|$$

$$< \left(\frac{1}{\omega}(1-\lambda) - 1\right) |a_{ii}| + \lambda |a_{ii}|$$

$(\because A \text{ is SDD})$

$$= (1-\lambda)\left(1 - \frac{1}{\omega}\right) |a_{ii}|$$

$$< 0$$

Hence both cases contradicts $(*)$; i.e. $-1 < \eta < 1$,

or $\rho(B) < 1$.

7 (a) $x^{k+1} = (\frac{1}{\omega}D+L)^{-1}(\frac{1}{\omega}D-(D+U))x^k + (\frac{1}{\omega}D+L)^{-1}b.$

(b). By Gershgorin Thm, any eigenvalue of A,

$$\lambda \in \bigcup_{i=1}^{3} \bar{B}_{a_{ii}}\left(\sum_{i\neq j}|a_{ij}|\right) = [1,7]\cup[0,8]\cup[3,5]$$
$$= [0,8].$$

but $\det A \neq 0$. Hence $\lambda \in (0,8]$.

Hence A is SPD. SOR converges $\iff 0 < \omega < 2$.

(c). We compute $\rho(M_J) = 10/16$

Hence $\omega_{opt} = \frac{2}{1+\sqrt{1-10/16}} = \frac{\cancel{4}\,8}{4+\sqrt{6}}$.

8 (a). Let D, L, U be the diagonal, lower triangular and upper triangular part of the ~~mat~~ A.

Note that $M_J = M_{GS} = -D^{-1}U$,
diagonal of $-D^{-1}U$ is all zero.
Hence eigenvalue of M_J and M_{GS} are all zero.
Therefore $\rho(M_J) = \rho(M_{GS}) = 0 < 1$, and
both methods anverge.

(b) $\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(M_J)^2}} = 1$

(c) Let $\tilde{D}$, $\tilde{L}$, $\tilde{U}$ be the diagonal, lower triangular, upper triangular part of A^T.

Note that $M_J = -\tilde{D}^{-1}(\tilde{L}+\tilde{U}) = -\tilde{D}^{-1}(\tilde{L})$

diagonal of $-\tilde{D}^{-1}\tilde{L}$ is all zero.

$$M_{GS} = -(\tilde{D}+\tilde{L})^{-1}\tilde{U} = -(\tilde{D}+\tilde{L})^{-1}\cdot 0 = 0$$

Hence $\rho(M_J) = \rho(M_{GS}) = 0$.

Both Jacobi and G-S. method converge.

9. $\det(A-\lambda I) = -(\lambda+a-1)(\lambda^2 - (a+2)\lambda - a^2+1)$

$\det(A-\lambda I) = 0 \Rightarrow \lambda = 1-a$ or $\dfrac{-(a+2) \pm \sqrt{(a+2)^2 - 4(a^2+1)}}{2}$

A is positive definite $\Leftrightarrow$ all eigenvalue > 0

$\Leftrightarrow$ $1-a > 0$ and $\dfrac{-(a+2) \pm \sqrt{(a+1)^2 - 4(a^2+1)}}{2} > 0$

$\Leftrightarrow$ $-1 < a \leq -\frac{4}{5}$ ~~and~~ or $0 \leq a < 1$

~~M_J~~ $\det(M_J - \tilde{\lambda} I) = -(\tilde{\lambda}+a)(\tilde{\lambda}^2 - a\tilde{\lambda} - (a^2+a))$

$\det(M_J - \tilde{\lambda} I) = 0 \Rightarrow \tilde{\lambda} = -a$ or $\dfrac{a \pm \sqrt{a(5a+4)}}{2}$

Jacobi converges $\Leftrightarrow$ $|\tilde{\lambda}| < 1$ for all possible $\tilde{\lambda}$

$\Leftrightarrow$ $-1 < a < 1$ and ($0 \leq a \leq \sqrt{2}-1$ or $a \leq -4/5$) and ($a \geq 0$ or $-1-\sqrt{2} < a \leq -4/5$)

$\Leftrightarrow$ $-1 < a \leq -4/5$ or $0 \leq a < \sqrt{2}-1$

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10 (a) if $0 < w < 2/\beta$, for any $1 \le k \le n$

$-1 < 1 - w\beta < 1 - w\lambda_k < 1 - w\alpha < 1$

Hence $\rho(1 - wA) = \max_{1 \le k \le n} |1 - w\lambda_k| < 1$

(b) $\because$ $\rho(1 - wA) = \max_{1 \le k \le n} |1 - w\lambda_k| = |1 - w\lambda_1|$

$= |1 - \lambda_1/\lambda_1| = 0$

Hence the iterative method converges in n iterations.

11. (a) $x^1 = Ax^0 = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$ $\tilde{x}^2 = \dfrac{Ax^1}{\|x^1\|_\infty} = \begin{pmatrix} 9/5 \\ 9/5 \\ 9/5 \end{pmatrix}$

So $x^2 = \tilde{x}^2 / \|\tilde{x}^2\|_\infty$

$= (1, 1, 1)^T = x^0.$

This method does not converges

Since $\lambda_1 = 3$ and $\lambda_2 = -3$

$|\lambda_1| = |\lambda_2|$.

So the power method does not converge.

(b) $A^2 x^0 = (9, 9, 9)^T = 9x^0.$

The power method works in this case.

Let $B = (v_1\ v_2\ v_3)$

12 $w_1 = u_1 = (2, -1, 0)^T / \|(2, -1, 0)^T\| = \frac{1}{\sqrt{5}}(2, -1, 0)^T$

$w_2 = v_2 - \langle v_2, u_1 \rangle u_1 = (1/5, 2/5, -1)^T$

$u_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{\sqrt{5}}{\sqrt{6}}(1/5, 2/5, -1)^T$

$w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2 = (8/3, 16/3, 8/3)^T$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{\sqrt{3}}{\sqrt{128}}\left(8/3,\ 16/3,\ 8/3\right)^T$$

$$= \frac{1}{\sqrt{6}}\left(1,\ 2,\ 1\right)^T$$

Hence

$$B = \begin{pmatrix} 2/\sqrt{5} & \sqrt{5}/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{5} & \frac{2\sqrt{5}}{\sqrt{6}} & 2/\sqrt{6} \\ 0 & \frac{-\sqrt{5}}{\sqrt{6}} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} \sqrt{5} & \sqrt{\frac{4}{5}} & \frac{1}{\sqrt{5}} \\ 0 & \sqrt{6/5} & \sqrt{242/15} \\ 0 & 0 & 16/\sqrt{6} \end{pmatrix}$$

13. $R(v, A) := \dfrac{v^* A v}{v^* v}$

If A is S.P.D., then $\lambda_n \le R(v, A) \le \lambda_1$.

$\lambda_1 \ge \dots \ge \lambda_n > 0$ are eigenvalues of A.

Let v_k be the eigenvector of A corresponding to λ_k.

For any $v \in \mathbb{R}^n$, write $v = \sum_{i=1}^{n} c_i v_i$

$$R(v, A) = \frac{v^* A v}{v^* v} = \frac{\sum_{i=1}^{n} \lambda_i c_i^* c_i v_i^* v_i}{\sum_{i=1}^{n} c_i^* c_i v_i^* v_i}$$

Since $\lambda_1 \le \lambda_i \le \lambda_n$,

$$\lambda_n \frac{\sum_{i=1}^{n} c_i^* c_i v_i^* v_i}{\sum_{i=1}^{n} c_i^* c_i v_i^* v_i} \le R(v, A) \le \lambda_1 \frac{\sum_{i=1}^{n} c_i^* c_i v_i^* v_i}{\sum_{i=1}^{n} c_i^* c_i v_i^* v_i}$$

i.e. $\lambda_n \le R(v, A) \le \lambda_1$.

<u>RQI</u>: Let $A \in M_{n\times n}(\mathbb{R})$,

Input x_0 s.t. $x_0^* x_0 = 1$, $\mu_0 = R(x_0, A)$

For $j = 0, 1, \dots$

Solve $(A - \mu_j I) z_{j+1} = x_j$

Let $x_{j+1} = z_{j+1} / \|z_{j+1}\|_2$

$\mu_{j+1} = R(x_{j+1}, A)$

End

Output x_{j+1}, μ_{j+1}.

14. Inverse power method:

Input x_0, $\|x_0\|_\infty = 1$

For $j = 0, 1, \ldots$

Solve $A z_{j+1} = x_j$

Let $x_{j+1} = z_{j+1} / \|z_{j+1}\|_\infty$

End

Output x_{j+1}, $\|A x_{j+1}\|_\infty$

$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\|x_0\|_\infty = 1$.

Solve $A z_1 = x_0 \Rightarrow z_1 = \begin{pmatrix} -2.5 \\ 2 \end{pmatrix}$

Let $x_1 = z_1 / \|z_1\|_\infty = (1, -0.8)^T$

Solve $A z_2 = x_1 \Rightarrow z_2 = \begin{pmatrix} 3.7 \\ -2.8 \end{pmatrix}$

Let $x_2 = z_2 / \|z_2\|_\infty = (1, -0.7568)^T$

$|\lambda| \approx \|A x_2\|_\infty = 0.2703$

15. Inverse power method with shift:

Input: x_0 s.t. $\|x_0\|_\infty = 1$

For $j = 0, 1, \ldots$

Solve $(A - \mu I) z_{j+1} = x_j$.

Let $x_{j+1} = z_{j+1} / \|z_{j+1}\|_\infty$

End

Output: x_{j+1}, $\|(A - \mu I) x_{j+1}\|_\infty$

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Let $x_0 = (1, 0, 0)^T$, $\mu = 1.3$

Solve $(A - \mu I) z_1 = x_0 \Rightarrow z_1 = (-19.20, 14.44, -5.39)^T$

Let $x_1 = z_1 / \|z_1\|_\infty = (-1, 0.7521, -0.2786)^T$

Solve $(A - \mu I) z_2 = x_1 \Rightarrow z_2 = (31.55, -23.08, 8.45)^T$

Let $x_2 = z_2 / \|z_2\|_\infty = (1, -0.7317, 0.2677)^T$

$(A - \mu I) x_2 = (-0.0317, 0.0238, -0.0088)^T$

$\lambda \approx -0.0317 + \mu = 1.2683$

16. Apply power method to A^{-1}.

17. (a), (b) : skipped.

(c) Iteration in (a) converges but for (b) does not converges.

For A , $|\lambda_1| > |\lambda_2|$.

B , $|\lambda_1| = |\lambda_2|$.

18. (a). $\| A^k (a_1 v_1 + a_2 v_2) \|_\infty = \| a_1 \lambda_1^k (v_1 + (-1)^k v_2) \|_\infty$

i.e. the Power method does not conveges in general

(b). Suppose $\lambda_1 = \lambda_2 = -\lambda_3$. From (a), the power method does not converge.

Suppose $\lambda_1 = \lambda_2$, $|\lambda_2| > |\lambda_3|$.

Let $x = a_1 v_1 + \dots + a_n v_n$

$$A^k x = A^k (a_1 v_1 + \dots + a_n v_n) = \lambda_1^k \left(a_1 v_1 + a_2 v_2 + \sum_{j=3}^{n} \frac{\lambda_j^k}{\lambda_1^k} a_j v_j \right) \rightarrow \lambda_1.$$

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19. Let $v_1 \cdots v_n$ be normalized eigenvectors w.r.t. $\lambda_1 \cdots \lambda_n$.

let $x^{(0)} = \sum_{i=1}^{n} c_i \cdot v_i$, $c_1 \neq 0$.

$$x^{(k)} = A x^{(k-1)} = \cdots = A^{(k)} x^{(0)}$$

$$= \sum_{i=1}^{n} c_i A^k v_i = \sum_{i=1}^{n} c_i \lambda_i^k v_i.$$

$$A x^{(k)} = \sum_{i=1}^{n} c_i \lambda_i^{k+1} v_i$$

$$A x^{(k)} \cdot x^{(k)} = \sum_{i=1}^{n} c_i^2 \lambda_i^{2k+1}$$

$$x^{(k)} \cdot x^{(k)} = \sum_{i=1}^{n} c_i^2 \lambda_i^{2k}$$

$$\frac{A x^{(k)} \cdot x^{(k)}}{x^{(k)} \cdot x^{(k)}} = \frac{\sum_{i=1}^{n} c_i^2 \cdot \lambda_i^{2k+1}}{\sum_{i=1}^{n} c_i^2 \lambda_i^{2k}} \leq \frac{\sum_{i=1}^{n} c_i^2 \lambda_i^{2k+1}}{c_1^2 \lambda_1^{2k}}$$

$$= \lambda_1 \left(1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \cdot \left(\frac{\lambda_i}{\lambda_1}\right)^{2k+1}\right)$$

$$\leq \lambda_1 \left(1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left|\frac{\lambda_2}{\lambda_1}\right|^{2k+1}\right)$$

$$= \lambda_1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 |\lambda_2| \left|\frac{\lambda_2}{\lambda_1}\right|^{2k+1}$$

$$= \lambda_1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right).$$

20 (a)

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 4 & 6 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(b). $Ax = b$
$\Rightarrow QRx = b \Rightarrow Rx = Q^T b = \begin{pmatrix} 5 \\ -1 \\ 0 \\ 2 \end{pmatrix}$.

$$\Rightarrow x = \begin{pmatrix} -3/2 \\ 1/2 \\ 1 \\ 1 \end{pmatrix}$$

(c) $A' = RQ = \begin{pmatrix} 6 & -2 & 4 & 0 \\ -2 & -2 & 0 & 0 \\ 2 & 0 & 0 & -2 \\ 1 & -1 & -1 & -1 \end{pmatrix}$

21. (a). $A = QR = \begin{pmatrix} 3/5 & -16/25 \\ 0 & 3/25 \\ 4/5 & 12/25 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}$

Solve $R\vec{x} = Q^T\vec{b}$

$\vec{x} = \begin{pmatrix} \frac{397}{625} \\ \frac{89}{125} \end{pmatrix}$

(b). Note: $(12, 20, -9)^T \perp (-2, 3, 4)^T$ and $(3, 0, 4)^T$

Let $v_1 = (3, 0, 4)^T$, $v_2 = (-2, 3, 4)^T$

$u_1 = \frac{v_1}{\|v_1\|} = (3/5, 0, 4/5)^T$

$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$
$= (-16/5, 3, 12/5)^T$

$u_2 = \frac{w_2}{\|w_2\|} = (-16/25, 3/5, 12/25)^T$

$w_3 = (12, 20, -9)^T$

$u_3 = \frac{w_3}{\|w_3\|} = (12/25, 4/5, 9/25)^T$

$B = QR = \begin{pmatrix} 3/5 & -16/25 & 12/25 \\ 0 & 3/5 & 4/5 \\ 4/5 & 12/25 & 9/25 \end{pmatrix} \begin{pmatrix} 5 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 25 \end{pmatrix}$