THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 2
Due Date: October 14, 2022

1. Solve the following PDE using Spectral Method:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = u(1,t), & t \in [0,\infty) \\ u(x,0) = f(x), & x \in [0,1] \end{cases}$$

where

$$f(x) = \begin{cases} -x(2x-1), & \text{if } x \in [0, \frac{1}{2}] \\ 0, & \text{else} \end{cases}$$

2. Recall the definitions of discrete and inverse discrete Fourier Transform from the lecture notes: Given: $f_0, f_1, \ldots, f_{n-1} \in C$, the discrete Fourier transform is defined as

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-i\frac{2jk\pi}{n}}$$

for $k = 0, 1, 2, \ldots, n-1$. And the inverse discrete Fourier Transform:

$$f_j = \sum_{k=0}^{n-1} c_k e^{i\frac{2jk\pi}{n}}$$

for $j = 0, 1, 2, \ldots, n-1$.
Check that the inverse discrete Fourier Transform does recover the discrete Fourier Transform.

3. Let $f = \{f_i\}_{i=0}^{n-1}$ and $g = \{g_i\}_{i=0}^{n-1}$ be two sequences of points in C that are periodic. Define convolution by

$$(f * g)_i = \sum_{k=0}^{n-1} f_k g_{i-k}$$

Prove that for $k = 0, \ldots, n-1$

$$(\widehat{f * g})(k) = n\hat{f}(k)\hat{g}(k)$$

where $\hat{f} = \text{DFT}(f)$.

4. In addition to 1D DFT, we can also see an example that is 2D DFT. Consider this alternative definition for the DFT on $N \times N$ images:

$$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) e^{2\pi i \frac{mk+nl}{N}}$$

.

(a) Show that the inverse DFT (iDFT) is defined by

$$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m,n) e^{-2\pi i \frac{pm+qn}{N}}.$$

(b) Determine the matrix U used to calculate the DFT of an $N \times N$ image, i.e. $\hat{f} = UfU$.

(c) Show that U is unitary (that is, $UU^* = U^*U = I$, where U^* is the conjugate transpose of U).

5. Consider the differential equation:

$$(**)\qquad a\frac{d^2u}{dx^2}+b\frac{du}{dx}=f(x)\text{ for }x\in(0,2\pi),$$

where $a,b>0$. Assume u and f are periodically extended to R. Divide the interval $[0,2\pi]$ into n equal portions and let $x_j=\frac{2\pi j}{n}$ for $j=0,1,2,...,n-1$.

Let $\mathbf{u}=(u(x_0),u(x_1),...,u(x_{n-1}))^T$ and $\mathbf{f}=(f(x_0),f(x_1),...,f(x_{n-1}))^T$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n\times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j=\frac{u(x_{j+2})-u(x_{j-2})}{4h}\quad\text{and}\quad(\mathcal{D}_2\mathbf{u})_j=\frac{u(x_{j+4})-2u(x_j)+u(x_{j-4})}{16h^2}.$$

for $j=0,1,2,...,n-1$.

(a) Explain why the differential equation (**) can be discretized as:

$$(***)\quad a\mathcal{D}_2\mathbf{u}+b\mathcal{D}_1\mathbf{u}=\mathbf{f}.$$

In other words, explain why $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Prove that $\overrightarrow{e^{ikx}}:=(e^{ikx_0},e^{ikx_1},...,e^{ikx_{n-1}})^T$ is an eigenvector of both $\mathcal{D}_1$ and $\mathcal{D}_2$ for $k=0,1,2,...,n-1$. What are their corresponding eigenvalues? Please explain your answer with details.

(c) Show that $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ forms a basis for C^n.

(d) Let $\mathbf{u}=\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}$ and $\mathbf{f}=\sum_{k=0}^{n-1}\hat{f}_k\overrightarrow{e^{ikx}}$, where $\hat{u}_k,\hat{f}_k\in C$. If $\mathbf{u}$ satisfies (***), show that

$$(a\lambda_k^2+b\lambda_k)\hat{u}_k=\hat{f}_k\text{ where }\lambda_k=i\frac{\sin(2kh)}{2h},$$

for $k=0,1,2,...,n-1$. Please explain your answer with details.