# Lecture 9: Recap:

**Definition:** (Discrete Fourier Transform) Given $f_0, f_1, \dots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)}$$

for $k = 0, 1, 2, \dots, n-1$

The inverse discrete Fourier Transform recovers the original signal:

$$f_j = \sum_{k=0}^{n-1} c_k \, e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \dots, n-1$$

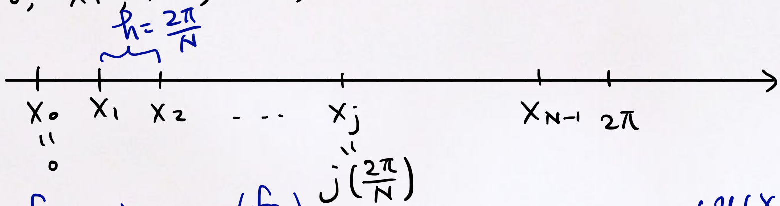
## Recall:

Consider: $\dfrac{d^2u}{dx^2} = f$ for $x \in [0, 2\pi]$ with periodic boundary condition.

$u(0) = u(2\pi)$

Suppose $f$ is measured only at $N$ discrete points:

$x_0, x_1, x_2, \dots, x_{N-1}$




(unknown)

Let $\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \in \mathbb{R}^N$ and $\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^N$

Then: $\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \approx \tilde{D}\vec{u}$ where $\tilde{D} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & 0 & \\ & 1 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$

$M_{N \times N}(\mathbb{R})$

(using the fact $u_0 = u_N$, $u_{-1} = u_{N-1}$)

Conclusion:

$$\frac{d^2u}{dx^2} = f \quad \text{can be discretized as} \quad \boxed{\tilde{D}\vec{u} = \vec{f}} \quad \text{(Linear System)}$$

Thus, $\frac{d^2u}{dx^2} = f$ can be discretized as $\boxed{\tilde{D}\vec{u} = \vec{f}}$ (Linear System)

Remark: Note that $\tilde{D}$ can be a very BIG matrix.

Goal: Design the numerical spectral method. We need to:

① Determine eigenvalues / eigenvectors of $\tilde{D}$

② Rank of $\tilde{D}$ (to understand the sol. of linear system)

In continuous case, $e^{ikx}$ is an eigenfunction of $\frac{d^2}{dx^2}$, that is periodic.

In discrete case, define:

$$\underset{\mathbb{C}^N}{\underset{\cap}{\overrightarrow{e^{ikx}}}} \stackrel{\text{def}}{=\!=} \begin{pmatrix} e^{ikx_0} \\ e^{ikx_1} \\ \vdots \\ e^{ikx_{N-1}} \end{pmatrix}$$

(Capture the values of $e^{ikx}$ at N discrete points)

Claim: $\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$ $(k = 0, 1, 2, \ldots, N-1)$

Mr precisely,

$$\tilde{D}\,\overrightarrow{e^{ikx}} = \underbrace{\left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)}_{= -\lambda_k^2}\overrightarrow{e^{ikx}}$$

where: $$\lambda_k^2 = \left(\frac{4\sin^2\frac{kh}{2}}{h^2}\right).$$

or $$\lambda_k = \left(\frac{2\sin\frac{kh}{2}}{h^2}\right)$$

<u>Claim:</u> $\left\{ \overrightarrow{e^{ikx}} \right\}_{k=0}^{N-1}$ is a basis of $\mathbb{C}^N$ (consisting of eigenvectors)

<u>Pf:</u>

$$\begin{pmatrix} | & | & & | \\ \overrightarrow{e^{i0x}} & \overrightarrow{e^{i(1)x}} & \cdots & \overrightarrow{e^{i(N-1)x}} \\ | & | & & | \end{pmatrix} = A_\omega = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & & & \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix}; \quad \omega = e^{i\frac{2\pi}{N}}$$

$$A_\omega \overline{A_\omega} = N I_{N\times N}$$

$$\Rightarrow \quad A_\omega^{-1} = \frac{1}{N}\overline{A_\omega}$$

# Numerical Spectral method

Since $\{ \overrightarrow{e^{ikx}} \}_{k=0}^{N-1}$ is a basis. We can write:

$$\vec{u} = \sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}} \quad \text{and} \quad \vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$\vec{u} \in \mathbb{C}^N$, $\hat{u}_k \in \mathbb{C}$, $\vec{f} \in \mathbb{C}^N$

In other words, for each $j$, $f_j = f(x_j) = \sum_{k=0}^{N-1} \hat{f}_k \left(\overrightarrow{e^{ikx}}\right)_j$

$$= \sum_{k=0}^{N-1} \hat{f}_k e^{ikx_j}, \quad x_j = \frac{2\pi j}{n}$$

(DFT!)

$\therefore$ $\hat{f}_k$ can be determined by DFT.

To solve $\dfrac{d^2u}{dx^2} = f$, we approximate it by

$$\tilde{D} \vec{u} = \vec{f}.$$

Now, $\tilde{D}\vec{u} = \vec{f}$ becomes:

$$\tilde{D}\left(\sum_{k=0}^{N-1} \hat{u}_k e^{\overrightarrow{ikx}}\right) = \sum_{k=0}^{N-1} \hat{f}_k e^{\overrightarrow{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k \underbrace{\tilde{D} e^{\overrightarrow{ikx}}}_{\substack{\| \\ (-\lambda_k^2) e^{\overrightarrow{ikx}}}} = \sum_{k=0}^{N-1} \hat{f}_k e^{\overrightarrow{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k (-\lambda_k^2) e^{\overrightarrow{ikx}} = \sum_{k=0}^{N-1} \hat{f}_k e^{\overrightarrow{ikx}}$$

Comparing coefficients, we get

$$\underbrace{-\lambda_k^2}_{\text{known}} \underbrace{\hat{u}_k}_{\text{unknown}} = \underbrace{\hat{f}_k}_{\text{known}} \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

(algebraic equation)

Thm: Let $A$ be a $n \times n$ complex matrix with eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, whose associated eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ forms a basis for $\mathbb{C}^n$.

Then: $A = QDQ^{-1}$ where

$$Q = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} ; \quad D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Also, if $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, then $\mathrm{Null}(A) = \mathrm{span}\{\vec{v}_1\}$.

$\mathrm{Rank}(A) = \mathrm{Rank}(D) =$ # of non-zero eigenvalues $= n-1$.

**Claim:** $\text{Rank}(\tilde{D}) = N-1$ and nullspace of $\tilde{D}$ is:

$$N(\tilde{D}) = \text{span}\left\{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\right\}.$$

**Proof:** $\text{Rank}(\tilde{D})$ = # of non-zero eigenvalues of $\tilde{D}$.

$$= \#\{-\lambda_1^2, -\lambda_2^2, \ldots, -\lambda_{N-1}^2\}$$

$$= N-1$$

$N(\tilde{D})$ = eigenspace of eigenvalue $= 0$ $\quad (\tilde{D}\vec{x} = \vec{0} = 0\vec{x})$

$$= \text{span}\left\{\begin{pmatrix}1\\\vdots\\1\end{pmatrix}\right\}$$

**Claim:** If $\vec{u}_1$ and $\vec{u}_2$ are both solutions of $\tilde{D}\vec{u} = \vec{f}$,

then: $\vec{u}_1 = \vec{u}_2 + c\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ for some const. $c$.

**Proof:**

$$\begin{array}{rl} & \tilde{D}\vec{u}_1 = \vec{f} \\ - & \tilde{D}\vec{u}_2 = \vec{f} \\ \hline & \tilde{D}(\vec{u}_1 - \vec{u}_2) = \vec{0} \end{array} \implies \vec{u}_1 - \vec{u}_2 \in N(\tilde{D})$$

$\therefore \vec{u}_1 = \vec{u}_2 + c\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ for some const. $c$

# Numerical spectral method

Since $\{\overrightarrow{e^{ikx}}\}_{k=0}^{N-1}$ is a basis of $\mathbb{C}^N$. We can write:

$$\underset{\in \mathbb{C}^N}{\vec{u}} = \sum_{k=0}^{N-1} \underset{\in \mathbb{C}}{\hat{u}_k} \overrightarrow{e^{ikx}}$$

and

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} = \underset{\in \mathbb{C}^N}{\vec{f}} = \sum_{k=0}^{N-1} \underset{\in \mathbb{C}}{\hat{f}_k} \overrightarrow{e^{ikx}}$$

$\Downarrow$

For each $j$, $f_j = \sum_{k=0}^{N-1} \hat{f}_k \left(\overrightarrow{e^{ikx}}\right)_j$

$$\Rightarrow \boxed{f_j = \sum_{k=0}^{N-1} \hat{f}_k e^{ikx_j}}$$

$$= \sum_{k=0}^{N-1} \hat{f}_k e^{ik\left(\frac{2\pi j}{N}\right)}$$

iDFT

$$\boxed{\hat{f}_k \text{ can be determined by DFT}}$$

Now, $\tilde{D}\vec{u} = \vec{f}$ becomes:

$$\tilde{D}\left(\sum_{k=0}^{N-1}\hat{u}_k \overrightarrow{e^{ikx}}\right) = \sum_{k=0}^{N-1}\hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \quad \sum_{k=0}^{N-1}\hat{u}_k \underbrace{\tilde{D}\left(\overrightarrow{e^{ikx}}\right)}_{=\,(-\lambda k^2)\overrightarrow{e^{ikx}}} = \sum_{k=0}^{N-1}\hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \quad \sum_{k=0}^{N-1}\hat{u}_k(-\lambda k^2)\overrightarrow{e^{ikx}} = \sum_{k=0}^{N-1}\hat{f}_k \overrightarrow{e^{ikx}}$$

Comparing coefficients: $\hat{u}_k(-\lambda k^2) = \hat{f}_k$ for $k=0,1,2,\dots,N-1$

(algebraic eqt)

$$(-\lambda_k^2)\,\hat{u}_k = \hat{f}_k \quad \text{for } k=0,1,2,\ldots,N-1$$

For $k=1,2,\ldots,N-1$, we have: $\hat{u}_k = \hat{f}_k / (-\lambda_k^2)$

For $k=0$, $\lambda_k = 0$.

We consider a special solution such that:

$$\hat{u}_0 = \frac{u_0 + u_1 + \ldots + u_{N-1}}{N} = 0$$

$$\hat{f}_0$$

$\therefore$ Set $\hat{u}_0 = 0$

Note that $\hat{f}_0 = -\lambda_0^2\,\hat{u}_0 = 0 \Rightarrow \dfrac{f_0 + f_1 + \ldots + f_{N-1}}{N} = 0$

$$\underbrace{\int_0^{2\pi} f(x)\,dx}_{\hat{f}_0} = \int_0^{2\pi} u''(x)\,dx = u'(x)\Big|_0^{2\pi} = 0 \quad (\text{periodic})$$

Once $\hat{u}_k$ are all defined for $k = 0, 1, 2, \ldots, N-1$.

$\vec{u}$ can be obtained:

$$\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ \vdots \\ u(x_{N-1}) \end{pmatrix} = \sum_{k=0}^{N-1} \hat{u}_k e^{ikx} \quad \text{(inverse DFT)}$$

$$\Leftrightarrow \quad \vec{u} = A_w \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \vdots \\ \hat{u}_k \end{pmatrix} \quad \text{where} \quad A_w = \begin{pmatrix} 1 & - & - & - & 1 \\ 1 & w & - & - & w^{N-1} \\ \vdots & & & & \\ 1 & w^{N-1} & - & - & w^{(N-1)^2} \end{pmatrix}$$

<u>Remark:</u> For any other sol $\vec{u}^*$ $(\tilde{D}\vec{u}^* = \vec{f})$,

$$\vec{u}^* = \underset{\text{(Special sol)}}{\vec{u}} + c\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$$

determined by the particular condition (boundary condition)