# Lecture 15:

## Recall:

- Want to solve: $A\vec{x} = \vec{f}$.

  Choose $N$ s.t. $A = N - P$.

  Splitting method: $N\vec{x}^{k+1} = P\vec{x}^{k} = \vec{f}$

- Jacobi: $N = D =$ diagonal part of $A$

  Gauss-Seidel: $N = L + D =$ lower part + diagonal part of $A$.

- $A$ is SDD $\Rightarrow$ Jacobi + Gauss-Seidel converge !!

**Note:** Previous example requires that $M = N^{-1}P$ is diagonalizable.

What if $N^{-1}P$ is NOT diagonalizable?

**Theorem:** Let $A \in M_{n\times n}(\mathbb{C})$ be a complex-valued matrix.

Then: $\lim_{k\to\infty} A^k = O$ if and only if $\rho(A) < 1$.

Simple consequence:

**Corollary:** The iterative scheme $\vec{x}^{k+1} = M\vec{x}^k + \vec{b}$ converges if and only if $\rho(M) < 1$.

**Proof:** Consider

$$\begin{aligned} \vec{x}^{k+1} &= M\vec{x}^k + \vec{b} \\ \vec{x}^* &= M\vec{x}^* + \vec{b} \\ \hline \therefore\ \vec{e}^{k+1} &= M\vec{e}^k \Rightarrow \vec{e}^k = M^k \vec{e}^0 \end{aligned}$$

$\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$.

$\vec{e}^k \to \vec{0}$ iff $M^k \to O$ iff $\rho(M) < 1$.

# Proof of Theorem

($\Rightarrow$) Let $\lambda$ be an eigenvalue of $A$ with eigenvector $\vec{v}$.

Then: $A^k \vec{v} = \lambda^k \vec{v}$. Thus, $\underbrace{\lim_{k\to\infty} A^k}_{0} \vec{v} = \lim_{k\to\infty} \lambda^k \vec{v}$

$$\Rightarrow \quad \vec{0} = \Big(\underbrace{\lim_{k\to\infty} \lambda^k}_{=\,0}\Big) \vec{v} \qquad \vec{v} \neq \vec{0}$$

$\therefore \quad |\lambda| < 1$ (for all eigenvalue $\lambda$)

$\therefore \quad \rho(A) < 1$

$= \max_j \{ |\lambda_j| : \lambda_j \text{ is eigenvalue} \}$

($\Leftarrow$) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $A$ (can be repeated)

We apply the idea of Jordan Canonical Form

$\Downarrow$

Find the invertible $Q$ such that $Q^{-1}AQ$ becomes

Simple

ALMOST looks like a diagonal

matrix.

Useful Fact: (Can be used without proof)

For any $A \in M_{n\times n}(\mathbb{C})$, there exists an invertible $Q \in M_{n\times n}(\mathbb{C})$ such that $A = QJQ^{-1}$ where $J$ is the Jordan Canonical Form of $A$. Actually,

$$J = \begin{pmatrix} J_{m_1}(\lambda_{i_1}) & & & O \\ & J_{m_2}(\lambda_{i_2}) & & \\ & & \ddots & \\ O & & & J_{m_s}(\lambda_{i_s}) \end{pmatrix}$$

($\lambda_{i_j}$ are eigenvalues of $A$)

where $J_{m_j}(\lambda_{i_j}) = \begin{pmatrix} \lambda_{i_j} & 1 & & O \\ & \lambda_{i_j} & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda_{i_j} \end{pmatrix} \in M_{m_j \times m_j}(\mathbb{C})$

$1 \le j \le s$

is called Jordan block with $\lambda_{i_j}$

e.g.

$$\bar{J} = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & O & O \\ O & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & O \\ O & O & \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} \end{pmatrix}$$

Now, $A^k = Q J^k Q^{-1}$ and

$$J^k = \begin{pmatrix} J^k_{m_1}(\lambda_{i_1}) & & \\ & \ddots & \\ & & J^k_{m_s}(\lambda_{i_s}) \end{pmatrix}$$

For $k \geq m_j - 1$

$$J^k_{m_j}(\lambda_{i_j}) = \begin{pmatrix} \lambda_{i_j}^k & C^k_1 \lambda_{i_j}^{k-1} & C^k_2 \lambda_{i_j}^{k-2} & \cdots & C^k_{m_j-1} \lambda_{i_j}^{k-m_j+1} \\ & \lambda_{i_j}^k & C^k_1 \lambda_{i_j}^{k-1} & \cdots & C^k_{m_j-2} \lambda_{i_j}^{k-m_j+2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & C^k_1 \lambda_{i_j}^{k-1} \\ & & & & \lambda_{i_j}^k \end{pmatrix} \quad \text{(By M.I.)}$$

$\because \rho(A) < 1 \quad \therefore \quad |\lambda_{i_j}| < 1$ for all $j$.

$$\therefore \lim_{k\to\infty} J_{m_j}^{k}(\lambda_{i_j}) = O \text{ for all } j$$

$$\text{and} \quad \lim_{k\to\infty} J^k = O$$

$$\therefore \lim_{k\to\infty} A^k = \lim_{k\to\infty} Q J^k Q^{-1} = O$$

## Splitting choice 3: Successive overrelaxation method (SOR)

Consider the iterative scheme:

( Suppose $A = L + D + U$ )

$$L\vec{x}^{k+1} + D\vec{y}^{k+1} + U\vec{x}^{k} = \vec{b} \quad \text{——} \quad (*)$$

$$\vec{x}^{k+1} = \vec{x}^{k} + \underset{\in\mathbb{R}}{\omega}\left(\vec{y}^{k+1} - \vec{x}^{k}\right) \quad \text{——} \quad (**)$$

$$\Longleftrightarrow \quad \vec{y}^{k+1} = \frac{1}{\omega}\left(\vec{x}^{k+1} + (\omega - 1)\vec{x}^{k}\right)$$

Putting (**) into (*):

$$\left(L + \frac{1}{\omega}D\right)\vec{x}^{k+1} + \frac{1}{\omega}\left(\omega U + (\omega - 1)D\right)\vec{x}^{k} = \vec{b}$$

or

$$\underbrace{\left(L + \frac{1}{\omega}D\right)}_{N}\vec{x}^{k+1} = \underbrace{\left(\frac{1}{\omega}D - (D+U)\right)}_{P}\vec{x}^{k} + \vec{b}$$

<u>Remark:</u>

- SOR is equivalent to:

$$A = N - P = \underbrace{\left(L + \frac{1}{\omega} D\right)}_{N} - \underbrace{\left(\frac{1}{\omega} D - (D+U)\right)}_{P}$$

$$N \vec{x}^{k+1} = P \vec{x}^{k} + \vec{b} \iff \vec{x}^{k+1} = N^{-1}P\, \vec{x}^{k} + N^{-1}\vec{b}$$

- $$\rho(N^{-1}P) = \rho\left(\underbrace{\left(L + \frac{1}{\omega} D\right)^{-1} \left(\frac{1}{\omega} D - (D+U)\right)}_{= M_\omega}\right)$$

has to be strictly less than 1 and it has to be small in order to converge fast.

- If $\omega = 1$, SOR = Gauss-Seidel

**Remark:** • SOR is equivalent to:

$$A = N - P = \underbrace{\left(L + \frac{1}{\omega}D\right)}_{N} - \underbrace{\left(\frac{1}{\omega}D - (D+U)\right)}_{P}$$

Or equivalently, $A = (a_{ij})$

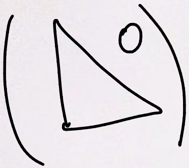
$$\begin{cases} a_{11} y_1^{k+1} + a_{12} x_2^k + \dots + a_{1n} x_n^k = b_1 & \text{for } x_1^{k+1} = x_1^k + \omega(y_1^{k+1} - x_1^k) \\ a_{21} x_1^{k+1} + a_{22} y_2^{k+1} + a_{23} x_3^k + \dots + a_{2n} x_n^k = b_2 & \text{for } x_2^{k+1} = x_2^k + \omega(y_2^{k+1} - x_2^k) \\ \vdots & \\ a_{n1} x_1^{k+1} + a_{n2} x_2^{k+1} + \dots + a_{nn} y_n^{k+1} = b_n & \text{for } x_n^{k+1} = x_n^k + \omega(y_n^{k+1} - x_n^k) \end{cases}$$

• SOR = Gauss-Seidel if $\omega = 1$.

## Condition for convergence

**Theorem:** The neccessary condition (not sufficient) for SOR to converge is $0 < \omega < 2$.

**Proof:** Consider: $\det(N^{-1}P)$ = product of eigenvalues of $N^{-1}P$



$$
\begin{aligned}
&= \det\left(\left(L + \tfrac{1}{\omega} D\right)^{-1} \left(\tfrac{1}{\omega} D - (D+U)\right)\right) \\
&= \det\left(\left(\tfrac{1}{\omega} D\right)^{-1}\right) \det\left(\tfrac{1}{\omega} D - D\right) \\
&= \det\left(\cancel{\omega}\, \cancel{D^{-1}}\right) \det\left(\tfrac{1}{\cancel{\omega}}(1-\omega)\cancel{D}\right) \\
&= \det\left((1-\omega) I\right) = (1-\omega)^n
\end{aligned}
$$

$$\therefore \left|\det(N^{-1}P)\right| = \prod_{i=1}^{n} |\lambda_i| = |1-\omega|^n$$

$\lambda_i$ = eigenvalue of $N^{-1}P$

$$|1-\omega|^n = \prod_{i=1}^{n} |\lambda_i| \le \left(\max_i |\lambda_i|\right)^n = \rho(N^{-1}P)^n$$

$$\rho(N^{-1}P) \geq |1-\omega|$$

Now, SOR converges iff $\rho(N^{-1}P) < 1$

$$\therefore \; |1-\omega| \leq \rho(N^{-1}P) < 1 \;\Rightarrow\; 0 < \omega < 2.$$

<u>Remark:</u> In general, the convergence and convergence rate depends on:

$$\rho(N^{-1}P) < 1$$

**Example:** Go back: $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix} := \vec{b}$.

**Recall:** $\underset{N_J^{-1} P_J}{\underset{\shortparallel}{\rho(M_J)}} = \frac{1}{10}$ ; $\underset{N_{GS}^{-1} P_{GS}}{\underset{\shortparallel}{\rho(M_{G\text{-}S})}} = \frac{1}{100}$

For SOR, $M_{SOR} = \left(L + \frac{1}{\omega} D\right)^{-1}\left(\frac{1}{\omega} D - (D+U)\right)$

$$= \left(\frac{1}{\omega}(D+\omega L)\right)^{-1}\left(\frac{1}{\omega}(D - \omega(D+U)\right)$$

$$= \cancel{\omega}\,(D+\omega L)^{-1}\,\cancel{\left(\frac{1}{\omega}\right)}\,((1-\omega)D - \omega U)$$

$$=$$

$\therefore$ $M_{SOR} = \begin{pmatrix} 1-\omega & -\omega/10 \\ -\frac{\omega(1-\omega)}{10} & \frac{\omega^2}{100} + (1-\omega) \end{pmatrix}$

$\therefore$ Characteristic polynomial of $M_{SOR}$:

$$\left[(1-\omega)-\lambda\right]\left[\frac{\omega^2}{100}+1-\omega-\lambda\right]-\frac{\omega^2(1-\omega)}{100}=0 \quad (\star\star)$$

Solving it:

$$\lambda=(1-\omega)+\frac{\omega^2}{200}\pm\frac{\omega}{20}\sqrt{4(1-\omega)+\frac{\omega^2}{100}}$$

Remark:

- Adjusting $\omega$ gives different eigenvalues and different $\rho(M_{SOR})$
- Put $\omega=1$ (G-S), $\lambda=0$, $\lambda=\frac{1}{100}$
- Choose $\omega$ such that $(\star\star)$ has a repeated root. That is,

$$4(1-\omega)+\frac{\omega^2}{100}=0 \Rightarrow \omega=1.002512579$$

Then: $\lambda=(1-\omega)+\frac{\omega^2}{200}=(1-\omega)-2(1-\omega)=\omega-1=0.002512579$

$\rho(M_J)=0.1$, $\rho(M_{G-S})=0.01$, $\rho(M_{SOR})=0.002512579$