Lecture 14:

Theorem: The Jacobi method converges if A is SDD. (Solve: $A\vec{x}=\vec{f}$)

Proof: Note: $\cancel{a_{ii}}\, x_i^{m+1} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^m + \frac{f_i}{a_{ii}}$ $(*)$ for $i=1,2,\ldots,n$.

Let $\vec{x}^*$ = sol of $A\vec{x}=\vec{f}$, we have:

$$x_i^* = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^* + \frac{f_i}{a_{ii}} \quad (**) \text{ for } i=1,2,\ldots,n$$

$(*) - (**)$
$$e_i^{m+1} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} e_j^m$$

$$\therefore |e_i^{m+1}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} |e_j^m| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} \|\vec{e}^m\|_\infty \leq r \|\vec{e}^m\|_\infty \text{ for all } i$$

where $r = \max_i \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} \right\} < 1$ $\quad$ $\left(\|\vec{e}^m\|_\infty \overset{\text{def}}{=} \max_j \{|e_j^m|\}\right)$

$\therefore \quad \|\vec{e}^{\,m+1}\|_\infty \le \overset{<1}{r}\, \|\vec{e}^{\,m}\|_\infty$

$\therefore \quad \underset{\substack{\downarrow \text{ as } m\to\infty \\ \vec{0}}}{\|\vec{e}^{\,m}\|_\infty} \le r^m \|\vec{e}^{\,0}\|_\infty \longrightarrow 0 \quad \text{as } m \to \infty$

$\therefore$ Jacobi method converges.

<u>Theorem:</u> The Gauss-Seidel method converges to the solution of $A\vec{x} = \vec{f}$ if A is SDD.

<u>Proof:</u> Gauss-Seidel method can be written as:

$$x_i^{m+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{m+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{m} + \frac{f_i}{a_{ii}} \quad (*)$$

for $i = 1, 2, \ldots, n$

Let $\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$. Then:

$$x_i^* = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^* - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^* + \frac{f_i}{a_{ii}} \quad (**)$$

$(*) - (**)$:

$$e_i^{m+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{m+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{m}$$

We'll prove:

$$\|\vec{e}^{\,m+1}\|_\infty \le r \|\vec{e}^{\,m}\|_\infty \quad \text{where} \quad r = \max_i \left\{ \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} \right\} < 1$$

Induction on i:

When $i=1$, $\quad |e_1^{m+1}| \le \sum_{j=2}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| |e_j^{m}| \le \|\vec{e}^{\,m}\|_\infty \sum_{j=2}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|$

$$\le r\,\|\vec{e}^{\,m}\|_\infty$$

$\therefore$ the statement is true when $i=1$.

Assume $\quad |e_k^{m+1}| \le r\,\|\vec{e}^{\,m}\|_\infty \quad$ for $k=1,2,\dots,i-1$

Then: $\quad |e_i^{m+1}| \le \sum_{j=1}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right| \underbrace{\boxed{|e_j^{m+1}|}}_{\le\, r\|\vec{e}^{\,m}\|_\infty} + \sum_{j=i+1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \underbrace{\boxed{|e_j^{m}|}}_{\le\, \|\vec{e}^{\,m}\|_\infty}$

$$\le \underbrace{r\,\|\vec{e}^{\,m}\|_\infty}_{<\,\|\vec{e}^{\,m}\|_\infty} \sum_{j=1}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right| + \|\vec{e}^{\,m}\|_\infty \sum_{j=i+1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|$$

$$\le \|\vec{e}^{\,m}\|_\infty \underbrace{\sum_{\substack{j=1 \\ j\ne i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|}_{\le\, r} \le r\,\|\vec{e}^{\,m}\|_\infty$$

By M.I., $|e_i^{m+1}| \le r\|\vec{e}^{\,m}\|_\infty$ for $i=1,2,\dots,n$

$\therefore \quad \|\vec{e}^{\,m+1}\|_\infty \le r\|\vec{e}^{\,m}\|_\infty$

$\Rightarrow \quad \|\vec{e}^{\,m}\|_\infty \le r^m\|\vec{e}^{\,0}\|_\infty \to \vec{0}$ as $m \to \infty$

as $r < 1$.

$\therefore$ G-S. converges!!

Example: Consider $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix} = \vec{b}$

A is SDP. $\therefore$ Both Jacobi and G-S method converges.

For Jacobi method, $\vec{x}^{k+1} = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\vec{x}^{k} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Let $M_J = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/10 \\ -1/10 & 0 \end{pmatrix}$

Eigenvalues of M_J are $\lambda_1 = 1/10$ and $\lambda_2 = -1/10$. $\therefore$ M_J is diagonalizable.

$\therefore$ $\rho(M_J) = 1/10$. $\therefore$ $\|\vec{e}^{\,m}\|_\infty \le \left(1/10\right)^m K_J$ ← const depending on the initial error $\vec{e}^{\,0}$

Recall: $\vec{e}^{\,m} = \lambda_1^m \left(a_1 \vec{u}_1 + \sum_{i=2}^{n} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^m \vec{u}_i \right)$ where $\vec{e}^{\,0} = \sum_{i=1}^{n} a_i \vec{u}_i$

For Gauss-Seidel, $\vec{x}^{k+1} = \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \vec{x}^{k} + \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Let $M_{G-S} = \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/10 \\ 0 & 1/100 \end{pmatrix}$.

Eigenvalues of M_{G-S} are $\lambda_1 = 1/100$ and $\lambda_2 = 0$ $\therefore$ M_{G-S} " diagonalizable

$\therefore$ $\rho(M_{G-S}) = 1/100$ $\therefore$ $\|\vec{e}^{m}\|_\infty \leq \left(\frac{1}{100}\right)^m K_{G-S} \Leftarrow$ const. depending on initial error.

$\therefore$ G-S converges faster.

Remark: To reduce the error by a factor of 10^{-m}, we need about $k \geq \dfrac{m}{-\log_{10}(\rho(M))}$ iterations.

Jacobi: $k \geq \dfrac{m}{-\log_{10}(1/10)} = m$

G-S : $k \geq \dfrac{m}{-\log_{10}(1/100)} = \dfrac{m}{2}$

$\therefore$ G-S converges twice faster than the Jacobi