Lecture 13: Recall:

Goal: Develop iterative method : find a sequence $\vec{x}_0, \vec{x}_1, \ldots$ such that $\vec{x}_k \to \vec{x}^* =$ solution of $A\vec{x} = \vec{f}$ as $k \to \infty$.

Remark: We can stop when error is small enough.

Method: Splitting method

Consider a linear system $A\vec{x} = \vec{f}$ where $A \in M_{n\times n}$ (n is BIG)

Split A as follows : $A = N + (A - N) = N - \underbrace{(N-A)}_{P} = N - P$

Then: $A\vec{x} = \vec{f} \iff (N-P)\vec{x} = \vec{f} \iff N\vec{x} = P\vec{x} + \vec{f}$

Develop an iterative scheme as follows:

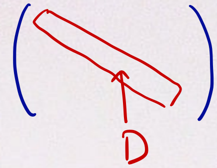
$$(*) \quad N\vec{x}^{n+1} = P\vec{x}^{n} + \vec{f}$$

If $\{\vec{x}^n\}_{n=1}^{\infty}$ converges, then it converges to the sol. $\vec{x}^*$ of $A\vec{x} = \vec{f}$

<u>Remark:</u>

- N must be simple enough so that $(*)$ can be solved easily
- Will $\{\vec{x}^n\}_{n=1}^{\infty}$ converge? Suitable N.

Splitting choice 1: Jacobi method

$A =$ 

Take $N = D$ = diagonal part of A

Split A as $A = D - (D-A)$

Then: $A\vec{x} = \vec{f} \iff D\vec{x} - (D-A)\vec{x} = \vec{f}$

$$\iff D\vec{x} = (D-A)\vec{x} + \vec{f}$$

We can consider an iterative scheme such that:

$$D\vec{x}^{k+1} = (D-A)\vec{x}^{k} + \vec{f}$$

$$\iff \vec{x}^{k+1} = D^{-1}(D-A)\vec{x}^{k} + D^{-1}\vec{f}$$

<u>Remark</u>: All diagonal entries of A must be non-zero, such that D is non-singular.

This is equivalent to solving: (assume $A = (a_{ij})_{1\le i,j\le n}$) $\vec{x}^k = \begin{pmatrix} x_1^k \\ \vdots \\ x_n^k \end{pmatrix}$

$$\begin{cases} a_{11}x_1^{k+1} + a_{12}x_2^k + a_{13}x_3^k + \dots + a_{1n}x_n^k = f_1 & (\text{for } x_1^{k+1}) \\ a_{21}x_1^k + a_{22}x_2^{k+1} + a_{23}x_3^k + \dots + a_{2n}x_n^k = f_2 & (\text{for } x_2^{k+1}) \\ \vdots \\ a_{n1}x_1^k + a_{n2}x_2^k + a_{n3}x_3^k + \dots + a_{nn}x_n^{k+1} = f_n & (\text{for } x_n^{k+1}) \end{cases}$$

Example: Consider: $\begin{pmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

Then: Jacobi method gives:

$$\vec{x}^{k+1} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 2 & -3 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}\vec{x}^k + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Start with $\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The sequence almost converge in 7 iteration to get: $\vec{x}_7 = \begin{pmatrix} -0.186 \\ 0.331 \\ -0.423 \end{pmatrix}$

Question:

- Does it always converge?
- Is the initialization important?

Splitting choice 2: Gauss - Seidel method

Split A as $A = L + D + U$

lower tri diagonal upper tri

Develop an iterative scheme : $L\vec{x}^{k+1} + D\vec{x}^{k+1} + U\vec{x}^{k} = \vec{f}$

(So, take $N = L + D$ and $P = -U$)

$\Leftrightarrow \quad \vec{x}^{k+1} = -(L+D)^{-1} U \vec{x}^{k} + (L+D)^{-1}\vec{f}$. (Solve lower-triangular linear system)

This is equivalent to:

$$\begin{cases} a_{11}x_1^{k+1} + a_{12}x_2^{k} + a_{13}x_3^{k} + \dots + a_{1n}x_n^{k} = f_1 & (\text{for } x_1^{k+1}) \\ a_{21}x_1^{k+1} + a_{22}x_2^{k+1} + a_{23}x_3^{k} + \dots + a_{2n}x_n^{k} = f_2 & (\text{for } x_2^{k+1}) \\ \vdots & \\ a_{n1}x_1^{k+1} + a_{n2}x_2^{k+1} + a_{n3}x_3^{k+1} + \dots + a_{nn}x_n^{k+1} = f_n & (\text{for } x_n^{k+1}) \end{cases}$$

Remark: Again, all diagonal entries of A must be non-zero, in order that $L+D$ is non-singular.

Example 2: Continue with Example 1.

$$\vec{x}^{k+1} = -\begin{pmatrix} 5 & 0 & 0 \\ -3 & 9 & 0 \\ 2 & -1 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{x}^k + \begin{pmatrix} 5 & 0 & 0 \\ -3 & 9 & 0 \\ 2 & -1 & 7 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Start with $\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. After 7 iterations, we get: $\vec{x}^7 = \begin{pmatrix} -0.186 \\ 0.331 \\ -0.423 \end{pmatrix}$

Question: Does Jacobi / Gauss-Seidel method always converge?

Example 3: Consider: $\begin{pmatrix} 1 & -5 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

Jacobi: $\vec{x}^{k+1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 5 \\ -7 & 0 \end{pmatrix} \vec{x}^k + \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1} \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

$\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ... $\vec{x}^7 = \begin{pmatrix} -214374 \\ -300127 \end{pmatrix}$ (Doesn't converge)

Gauss-Seidel also doesn't converge!!

Analysis of convergence

Let $A = N - P$

Goal: Solve $A\vec{x} = \vec{f} \iff (N-P)\vec{x} = \vec{f}$

Consider the iterative scheme: $N\vec{x}^{m+1} = P\vec{x}^m + \vec{f}$, $m = 0, 1, 2, \dots$

Let $\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$. Define error: $\vec{e}_m = \vec{x}^m - \vec{x}^*$, $m = 0, 1, 2 \dots$

Now, (1) $N\vec{x}^{m+1} = P\vec{x}^m + \vec{f}$

(2) $N\vec{x}^* = P\vec{x}^* + \vec{f}$ $\quad$ ($\because$ $A\vec{x}^* = \vec{f} \iff (N-P)\vec{x}^* = \vec{f}$)

(1)−(2): $N(\vec{x}^{m+1} - \vec{x}^*) = P(\vec{x}^m - \vec{x}^*)$

$$\iff N\vec{e}^{m+1} = P\vec{e}^m \iff \vec{e}^{m+1} = \overbrace{N^{-1}P}^{M}\vec{e}^m.$$

Let $M = N^{-1}P$. We get: $\vec{e}^m = M^m \vec{e}^0$

Assume a simple case: let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be the set of linearly independent eigenvectors of M ($\vec{u}_i$ can be complex-valued vectors)

(In other words, assume diagonalizable)

Let $\vec{e}^{\,0} = \sum_{i=1}^{n} a_i \vec{u}_i$. Then:

$$\vec{e}^{\,m} = M^m \vec{e}^{\,0} = \sum_{i=1}^{n} a_i M^m \vec{u}_i = \sum_{i=1}^{n} a_i \lambda_i^m \vec{u}_i$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are corresponding eigenvalues.

WLOG, we can assume:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda$$

Assume $|\lambda_1| < 1$. Then:

$$\vec{e}^{\,m} = \lambda_1^m \left\{ a_1 \vec{u}_1 + \sum_{i=2}^{n} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^m \vec{u}_i \right\} \to 0 \text{ as } m \to \infty$$

<u>Remark:</u> • In order to reduce error by a factor of 10^{-m}, then we need about k iterations such that $|\lambda_1|^k < 10^{-m}$

That is, $k > \dfrac{m}{-\log_{10}(\rho(M))} := \dfrac{m}{R}$

Here, we call $\rho(M) = |\lambda_1|$ the asymptotic convergence factor. or the spectral radius.

$\therefore$ $\rho(M) \overset{\text{def}}{:=} \max\limits_{k} \{ |\lambda_k| : \lambda_k = \text{eigenvalue of } M \}$ is a good indicator for convergence.

• Finding $\rho(M)$ is difficult $\Longrightarrow$ Numerically (next topic)

Useful Theorem for eigenvalues

Gerschgorin Theorem

Consider $\vec{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$ = eigenvector of $A = (a_{ij})_{1 \le i,j \le n}$ with eigenvalue λ.

$\in M_{n \times n}(\mathbb{C})$

Then: $A\vec{e} = \lambda \vec{e}$.

For $1 \le i \le n$, $\displaystyle\sum_{j=1}^{n} a_{ij} e_j = \lambda e_i$

$$\Leftrightarrow a_{ii} e_{ii} + \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} e_j = \lambda e_i$$

$$\Leftrightarrow (a_{ii} - \lambda) e_{ii} = - \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} e_j$$

$$\therefore |a_{ii} - \lambda| \, |e_i| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \, |e_j|$$

Let ℓ be the index such that $\underset{\neq 0}{|e_\ell|} \geq |e_j|$ for $\forall j$

Then: $$|a_{\ell\ell}-\lambda|\,\cancel{|e_\ell|} \leq \sum_{\substack{j=1\\ j\neq \ell}}^{n} |a_{\ell j}|\,|e_j| \leq \sum_{\substack{j=1\\ j\neq \ell}}^{n} |a_{\ell j}|\,\cancel{|e_\ell|}$$

$$\therefore\quad |a_{\ell\ell}-\lambda| \leq \sum_{\substack{j=1\\ j\neq \ell}}^{n} |a_{\ell j}| \quad\therefore\quad \lambda \in \overline{B_{a_{\ell\ell}}\Big(\sum_{\substack{j=1\\ j\neq \ell}}^{n} |a_{\ell j}|\Big)}$$

Ball of radius $\sum_{\substack{j=1\\ j\neq \ell}}^{n} |a_{\ell j}|$ centered at $a_{\ell\ell}$

Note: We don't know ℓ unless we know λ and $\vec{e}$.

BUT: we can conclude:

$$\lambda \in \overline{B_{a_{\ell\ell}}\Big(\sum_{\substack{j=1\\ j\neq \ell}}^{n} |a_{\ell j}|\Big)} \subseteq \bigcup_{i=1}^{n} \overline{B_{a_{ii}}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\Big)}$$

Example: Let $A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ 0 & & -1 & 2 \end{pmatrix}$

$\left(\text{Eigenvalues: } \begin{array}{l} \lambda_1 = 3.618 \\ \lambda_2 = 2.618 \\ \lambda_3 = 1.382 \\ \lambda_4 = 0.382 \end{array} \right)$

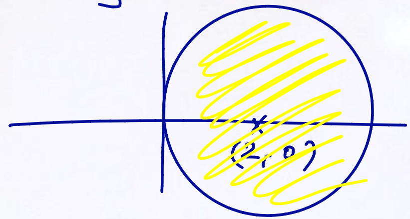
For $\ell = 1, 4$,

$$B_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j\neq \ell}}^{4} |a_{\ell j}|\right) = \{\lambda : |\lambda - 2| \le 1\}$$

For $\ell = 2, 3$,

$$B_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j\neq \ell}}^{4} |a_{\ell j}|\right) = \{\lambda : |\lambda - 2| \le 2\}$$

$$\therefore \bigcup_{i=1}^{4} B_{a_{ii}}\left(\sum_{\substack{j=1 \\ j\neq i}}^{4} |a_{ij}|\right) = \{\lambda : |\lambda - 2| \le 2\}$$



$\because$ A is symmetric, all eigenvalues are real,

$\therefore$ all eigenvalues are between 0 and 4

$\therefore$ $\rho(A) \le 4$

Condition for Jacobi / Gauss-Seidel to converge

Definition: A matrix $A = (a_{ij})_{1\le i,j\le n}$ is called strictly diagonally dominant (SDD) if $|a_{ii}| > \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}|$ for $i=1,2,\dots,n$

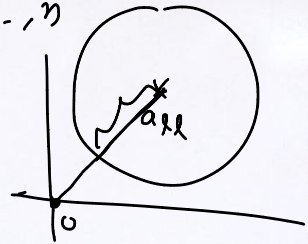
~~(SPD)~~

Theorem: If a matrix A is SDD, then A is non-singular.

Proof: All eigenvalues $\lambda \in \bigcup_{\ell=1}^{n} \overline{B}_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j\ne \ell}}^{n} |a_{\ell j}|\right)$

A is SDD iff $|a_{\ell\ell}| > \sum_{\substack{j=1 \\ j\ne \ell}}^{n} |a_{\ell j}|$ for $\ell=1,2,\dots,n$

$\therefore$ Every $\overline{B}_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j\ne \ell}}^{n} |a_{\ell j}|\right)$ must NOT contain 0.



$\therefore$ No eigenvalue is 0 and so A is non-singular.