THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 1 Suggested Solution

1. Solve the following ODE using method of integrating factor

$$x^4y' + 5x^3y = e^{-x}, \quad x < 0$$

with condition $y(-1) = 0$.

Solution:

$$\begin{aligned} x^4y' + 5x^3y &= e^{-x} \\ y' + \frac{5}{x}y &= \frac{e^{-x}}{x^4} \end{aligned}$$

Let $M(x) = e^{\int \frac{5}{x}\,dx} = |x|^5 = (-x)^5$ as $x < 0$.

$$\begin{aligned} M(x)\left(y' - \frac{5}{-x}y\right) &= M(x)\left(\frac{e^{-x}}{(-x)^4}\right) \\ (-x)^5y' - 5(-x)^4y &= \frac{e^{-x}(-x)^5}{(-x)^4} \\ \left(y(-x)^5\right)' &= -xe^{-x} \end{aligned}$$

Solving,

$$y(x) = -\frac{(x+1)e^{-x} + C}{x^5}$$

Putting initial condition,

$$y(x) = -\frac{(x+1)e^{-x}}{x^5}$$

2. Solve the following second order ODE using method of integrating factor

$$-2y'' + 4y = 8x^2 + 13x - 11$$

with conditions $y'(0) = 0$ and $y(1) = 4$.

Solution:

$$\begin{aligned} -2y_1'' + 4y_1 &= 0 \\ y_1'' &= 2y_1 \\ y_1' \cdot y_1'' &= 2y_1 \cdot y_1' \\ \left((y_1')^2\right)' &= 2(y_1^2)' \\ (y_1')^2 &= 2y_1^2 + C \end{aligned}$$

Suppose $C = 0$.

$$y_1' = \pm\sqrt{2}y_1$$

If $y_1' = \sqrt{2}y_1$, $y_1 = A_1e^{\sqrt{2}x}$.
If $y_1' = -\sqrt{2}y_1$, $y_1 = A_2e^{-\sqrt{2}x}$.
So, a general solution is

$$y_1 = A_1e^{\sqrt{2}x} + A_2e^{-\sqrt{2}x}$$

Let $y_2(x) = A_3x^2 + A_4x + A_5$. Putting y_2 into the differential equation,

$$y_2(x) = 2x^2 + \frac{13}{4}x - \frac{3}{4}$$

Combining y_1 and y_2,

$$y(x) = A_1 e^{\sqrt{2}x} + A_2 e^{-\sqrt{2}x} + 2x^2 + \frac{13}{4}x - \frac{3}{4}$$

Putting initial conditions,

$$A_1 = -\frac{4e^{\sqrt{2}} + 13\sqrt{2}}{8\big(e^{2\sqrt{2}} + 1\big)}, \quad A_2 = \frac{13\sqrt{2}e^{2\sqrt{2}} - 4e^{\sqrt{2}}}{8\big(e^{2\sqrt{2}} + 1\big)}$$

3. Please show that

$$\int_0^{2\pi} \cos kx \cos mx \, dx = \begin{cases} 2\pi, \text{ if } k = m = 0 \\ \pi, \text{ if } k = m \neq 0 \\ 0, \text{ if } k \neq m \end{cases}$$

and that

$$\int_0^{2\pi} \sin kx \sin mx \, dx = \begin{cases} 0, \text{ if } k = m = 0 \\ \pi, \text{ if } k = m \neq 0 \\ 0, \text{ if } k \neq m \end{cases}$$

where m, k are non-negative integer.

Solution:
For cosine terms,
if $k = m = 0$,

$$\int_0^{2\pi} dx = 2\pi$$

if $k = m \neq 0$,

$$\begin{aligned} \int_0^{2\pi} \cos^2 kx dx &= \int_0^{2\pi} \frac{\cos 2kx + 1}{2} dx \\ &= \pi + \frac{1}{4k}\big[\sin 2kx\big]_0^{2\pi} \\ &= \pi \end{aligned}$$

if $k = m = 0$,

$$\begin{aligned} \int_0^{2\pi} \cos kx \cos mx dx &= \frac{1}{2}\int_0^{2\pi} \big(\cos(k+m)x + \cos(k-m)x\big) dx \\ &= \frac{1}{2}\left[\frac{1}{k+m}\sin(k+m)x + \frac{1}{k-m}\sin(k-m)x\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

Sine terms are similar.

4. Let $f(x) = x^2$, then please compute the Fourier series of $f(x)$ on $[-1, 1]$.

Solution:
Scaling f from $[-1, 1]$ to $[-\pi, \pi]$ by $g(y) = f(\frac{y}{\pi}) = \frac{y^2}{\pi^2}$,
Computing the Fourier Series of g,

$$\begin{aligned} A_0 &= \frac{1}{3} \\ A_n &= \frac{4(-1)^n}{\pi^2 n^2} \\ B_n &= 0 \\ g(x) &= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos nx \end{aligned}$$

Changing back the variable,

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos \pi n x$$

5. Find the Fourier series solution to the differential equation

$$y'' + 2y = 3x$$

where $0 \le x \le 1$ and $y(0) = y(1) = 0$.

Solution:
Splitting the problem into homogeneous part and non-homogeneous part:

$$y_1'' + 2y_1 = 0 \quad (1)$$
$$y_2'' + 2y_2 = 3x \quad (2)$$

From (1), we can see that $y_1 = A\cos\sqrt{2}x + B\sin\sqrt{2}x$.
Considering (2), note the Fourier series of x on $[0, 2\pi]$ is given by:

$$\pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

and hence the Fourier Series of $3x$ on $[0, 1]$ is given by:

$$\frac{3}{2} - \sum_{n=1}^{\infty} \frac{3}{\pi n} \sin 2\pi n x$$

Assuming $y_2(x) = A_0 + \sum_{n=1}^{\infty}\big(A_n \cos 2\pi n x + B_n \sin 2\pi n x\big)$, and comparing the two sides of (2),

$$A_0 + \sum_{n=1}^{\infty}\Big((-4\pi^2 n^2 + 2)A_n \cos 2\pi n x + (-4\pi^2 n^2 + 2)B_n \sin 2\pi n x\Big) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{3}{\pi n} \sin 2\pi n x$$

we have:

$$A_0 = \frac{3}{2}$$
$$A_n = 0$$
$$B_n = -\frac{3}{\pi n(2 - 4\pi^2 n^2)}$$

Putting in boundary conditions:

$$\begin{cases} 0 = y(0) = y_1(0) + y_2(0) = A + \frac{3}{2} \\ 0 = y(1) = y_1(1) + y_2(1) = A\cos\sqrt{2} + B\sin\sqrt{2} + \frac{3}{2} \end{cases}$$

Solving, $A = -\frac{3}{2}$ and $B = \frac{3(\cos\sqrt{2} - 1)}{2\sin\sqrt{2}}$. So, we have

$$y(x) = -\frac{3}{2}\cos\sqrt{2}x + \frac{3(\cos\sqrt{2} - 1)}{2\sin\sqrt{2}}\sin\sqrt{2}x + \frac{3}{2} - \sum_{n=1}^{\infty} \frac{3}{\pi n(2 - 4\pi^2 n^2)} \sin 2\pi n x$$

6. Solve the following PDE using Fourier series

$$\begin{cases} u_t(t,x) = 4u_{xx}(t,x), \quad 0 < x < \pi, t > 0 \\ u_x(t,0) = 0 = u_x(t,\pi), \quad t > 0 \\ u(0,x) = f(x), \quad 0 \le x \le \pi \end{cases}$$

where $f(x) = x$.

Solution:
Let $v(t,x)$ be an even extension of u on x-coordinate. That is:

$$u(t,x) = \begin{cases} u(t,x), & \text{if } x \geq 0 \\ u(t,-x), & \text{if } x < 0 \end{cases}$$

Then, the PDE problem becomes:

$$\begin{cases} v_t(t,x) = 4v_{xx}(t,x), & -\pi < x < \pi, t > 0 \\ v_x(t,-\pi) = 0 = v_x(t,\pi), & t > 0 \\ v(0,x) = |x|, & -\pi \leq x \leq \pi \end{cases}$$

Let $v(t,x) = T(t)X(x)$. Then

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{4T(t)} = \lambda$$

for some $\lambda \in \mathbb{R}$. Solving the above equations with boundary conditions,

$$\begin{aligned} v_0(t,x) &= A_0 \\ v_n(t,x) &= A_n e^{-4n^2 t} \cos nx \\ v(t,x) &= \sum_{n=0}^{\infty} v_n(t,x) = A_0 + \sum_{n=1}^{\infty} A_n e^{-4n^2 t} \cos nx \end{aligned}$$

Note $v(0,x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx = |x|$ and the Fourier Series of $|x|$ is given by:

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\big((-1)^n - 1\big)}{\pi n^2} \cos nx$$

Hence, we have

$$v(t,x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\big((-1)^n - 1\big)}{\pi n^2} e^{-4n^2 t} \cos nx$$

The restriction of $v(t,x)$ on $[0,\pi]$ is our desired $u(t,x)$.