THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 5
Due Date: December 9 before 11:59 PM

1. Consider the linear system $A\boldsymbol{x} = \boldsymbol{k}$, where

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{pmatrix} \text{ and } \boldsymbol{k} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $\boldsymbol{x}^* = (1,1,1,1)^T$ be the solution of the linear system. Suppose $\{\boldsymbol{x}^{(m)}\}_{m=1}^{\infty}$ and $\{\boldsymbol{y}^{(m)}\}_{m=1}^{\infty}$ are the sequences of vectors obtained by the Jacobi method and Gauss-Seidel method respectively to solve the linear system with initialization $\boldsymbol{x}^{(0)} = \boldsymbol{y}^{(0)} = (0,0,0,0)^T$. Let $\boldsymbol{e}_J^{(m)} := \boldsymbol{x}^{(m)} - \boldsymbol{x}^*$ and $\boldsymbol{e}_{GS}^{(m)} := \boldsymbol{y}^{(m)} - \boldsymbol{x}^*$ be the error vectors at the m-th iteration for the Jacobi and Gauss-Seidel method respectively.

(a) Show that: $\boldsymbol{e}_J^{(m)} = -2^{-m}(1,1,1,1)^T$ for $m \geq 1$.

(b) Show that: $\boldsymbol{e}_{GS}^{(m)} = -4^{-m}(2,2,1,1)^T$ for $m \geq 1$.

(c) Show that $\|\boldsymbol{e}_{GS}^{(m)}\|_2 < \|\boldsymbol{e}_J^{(m)}\|_2$ for $m \geq 1$. Hence, the Gauss-Seidel method converges faster than the Jacobi method.

2. Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Consider:

$$f(\boldsymbol{\eta}) = \frac{1}{2}\boldsymbol{\eta}^T A\boldsymbol{\eta} - \boldsymbol{b}^T\boldsymbol{\eta}$$

(a) For an iterative scheme $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k\boldsymbol{p}_k$, where $\boldsymbol{p}_k$ is a fixed direction, find α_k such that $f(\boldsymbol{x}^{(k+1)})$ is minimized.

(b) The conjugate gradient method is given by

$$\begin{aligned} \boldsymbol{x}^{(k+1)} &= \boldsymbol{x}^{(k)} + \alpha_k\boldsymbol{p}_k, \\ \boldsymbol{p}_{k+1} &= -\boldsymbol{r}_{k+1} - \beta_k\boldsymbol{p}_k, \\ \beta_k &= -\frac{\langle \boldsymbol{r}_{k+1}, \boldsymbol{p}_k\rangle_A}{\langle \boldsymbol{p}_k, \boldsymbol{p}_k\rangle_A} \end{aligned}$$

where α_k is in the form given by (a), $\boldsymbol{r}_k = A\boldsymbol{x}^{(k)} - \boldsymbol{b}$, $\boldsymbol{p}_0 = -\boldsymbol{r}_0$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_A = \boldsymbol{x} \cdot A\boldsymbol{y}$, $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ is an arbitray initial guess. Provided that $\boldsymbol{r}_i \cdot \boldsymbol{r}_j = 0$ and $\langle \boldsymbol{p}_i, \boldsymbol{p}_j\rangle_A = 0$ for all $i \neq j$, show that $\beta_k = -\frac{\boldsymbol{r}_{k+1}\cdot\boldsymbol{r}_{k+1}}{\boldsymbol{r}_k\cdot\boldsymbol{r}_k}$.

3. Consider the gradient descent method for solving $A\boldsymbol{x} = \boldsymbol{b}$ with some $\alpha \in \mathbb{R}$ and A to be a symmetric positive definite matrix:

$$\begin{aligned} \boldsymbol{x}^{k+1} &= \boldsymbol{x}^k + \alpha\boldsymbol{d}^k \\ \boldsymbol{d}^k &= -(A\boldsymbol{x}^k - \boldsymbol{b}) \end{aligned}$$

Prove that the method converges if and only if $\alpha < \frac{2}{\lambda_j}$ for all j where λ_j are the eigenvalues of A.

Hint: suppose $\boldsymbol{\eta}$ is the solution to $A\boldsymbol{x} = \boldsymbol{b}$, them η satisfies

$$\boldsymbol{\eta} = \boldsymbol{\eta} + \alpha(A\boldsymbol{\eta} - \boldsymbol{b}).$$

Using this equation, start with the error vector $\boldsymbol{e}^k = \boldsymbol{x}^k - \boldsymbol{\eta}$ to make some observation.