THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 4
Due Date: November 25 before 11:59 PM

1. Consider a $n \times n$ tridiagonal linear system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{pmatrix} \alpha & -\beta & & \\ -\beta & \alpha & -\beta & \dots \\ & \ddots & \ddots & \ddots \\ & & -\beta & \alpha \end{pmatrix}$$

where $\alpha \geq \beta$.

(a) Prove that the eigenvectors of A are given by

$$q_j = \begin{pmatrix} \sin(j\theta) \\ \sin(2j\theta) \\ \vdots \\ \sin(nj\theta) \end{pmatrix}$$

for $j = 1, 2, \dots, n$ and $\theta = \frac{\pi}{n+1}$.

(b) Suppose $\alpha = 2$ and $\beta = 1$. Prove that the Jacobi method to solve $A\mathbf{x} = \mathbf{b}$ converges by looking at the spectral radius of a suitable matrix. Please explain your answer with details.

(c) Suppose $\alpha = 2$ and $\beta = 1$. Using the Housholder-John theorem, prove that the Gauss-Seidel method to solve $A\mathbf{x} = \mathbf{b}$ converges. Please explain with details.

(d) Suppose $\alpha = 2$ and $\beta = 1$. Explain why the SOR method converges for $0 < \omega < 2$. What is the optimal parameter ω_{opt} in the SOR method to obtain the fastest convergence. Please explain your answer with details.

2. Consider:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Find the QR factorization of A by Gram-Schmidt process. Compute the first iteration in QR method. Please show all your steps.

3. Consider:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Suppose an initial vector is given as $\mathbf{x}^{(0)} = (1, -1, 1)^T$. Calculate the first iteration of power method. Find the eigenvalue and the normalised eigenvector associated to it.

4. Let $A \in M_{n\times n}(\mathbb{C})$ be a $n \times n$ complex-valued matrix. Suppose the characteristic polynomial of A is given by: $f_A(t) = (-1)^n(t-\lambda_1)(t-\lambda_2)...(t-\lambda_n)$, where $\lambda_1, ..., \lambda_n$ are eigenvalues of A. Assuming that

$$|\lambda_1| = |\lambda_2| = ... = |\lambda_k| > |\lambda_{k+1}| \geq ... \geq |\lambda_n|,$$

where $k < n$. Suppose $A = QJQ^{-1}$, where J is the Jordan canonical form of A and Q is an invertible matrix. Assuming that the diagonal entries of J are arranged in descending order in terms of their magnitudes. Denote the j-th column of Q by $\mathbf{q}_j$, where $\mathbf{q}_1$, $\mathbf{q}_2$, ..., $\mathbf{q}_k$ are eigenvectors of A associated to λ_1, λ_2,..., λ_k respectively.

Let $\mathbf{x}_0$ be the initial vector defined as $\mathbf{x}_0 = a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + ... + a_n\mathbf{q}_n$, where $a_j \in \mathbb{C}$ for $1 \leq j \leq n$ and $a_i \neq 0$ for $i = 1, 2, ..., k$. Consider the iterative scheme:

$$\mathbf{x}_{j+1} = \frac{A\mathbf{x}_j}{||A\mathbf{x}_j||_\infty} \text{ for } j = 0, 1, 2, ...$$

(a) Suppose $\lambda_1 = \lambda_2 = ... = \lambda_k \in \mathbb{R}$. will $||A\mathbf{x}_j||_\infty$ always converge as $j \to \infty$. If yes, what will it converge to? If not, please give a counter-example and explain your answer with details. Please show the full details of your proof.

(b) In general, if $|\lambda_1| = |\lambda_2| = ... = |\lambda_k|$, will $||A\mathbf{x}_j||_\infty$ always converge $j \to \infty$? If yes, what will it converge to? If not, please give a counter-example and explain your answer with details. Please show the full details of your proof.