THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 3
Due Date: November 10 before 11:59PM

1. Consider the following system of equations:

$$\begin{aligned} -3x + 3y - 6z &= 4 \\ -4x + 7y - 8z &= 8 \\ 5x + 7y - 9z &= 12 \end{aligned}$$

   (a) Determine whether the Jacobi method converges.
   (b) Using initial approximation $x^{(0)} = (1, 0, 0)^T$, conduct the first two Jacobi iterations.

2. Consider the following system of equations:

$$\begin{aligned} -3x + 3y - 6z &= 4 \\ -4x + 7y - 8z &= 8 \\ 2x + 7y - 9z &= 12 \end{aligned}$$

   (a) Determine whether the Gauss-Seidel method converges.
   (b) Using initial approximation $x^{(0)} = (1, 1, 1)^T$, conduct the first two Gauss-Seidel iterations.

3. Consider the following system of equations:

$$\begin{aligned} -3x - 2y - z &= 1 \\ -4x + 4y - 6z &= 2 \\ -2x - 3y + 5z &= 3 \end{aligned}$$

   (a) Determine whether the SOR method converges if $\omega = 1.2$.
   (b) Determine whether the SOR method converges if $\omega = 1.4$.
   (c) Using initial approximation $x^{(0)} = (0, 0, -1)^T$, conduct the first two SOR iterations where $\omega = 1.2$.

4. Recall in Homework 2, we discussed an alternative definition for 2D DFT. Here, we introduce a more natural definition for 2D DFT. What 2D DFT does is actually applying DFT horizontally or vertically, and then apply DFT on the other direction.
Let $F \in \mathbb{C}^{N \times N}$. We define 2D DFT as

$$\hat{F}(m, n) = DFT(F)(m, n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k, l) e^{-2\pi i \frac{mk+nl}{N}}$$

   (a) Recall 1D DFT is given by $\hat{f} = \frac{1}{N} \overline{A_\omega} f$ where $f \in \mathbb{C}^n$ is a column vector.
   By applying DFT on each row of F, and second DFT on each column, show that the 2D DFT of F is can be written as

$$\hat{F} = \frac{1}{N^2} \overline{A_\omega} F \overline{A_\omega}$$

   (b) Given the computation cost for 1D FFT is of $O(N \log(N))$. By applying FFT in above approach, we can get 2D FFT.
   What is the computation cost for 2D FFT?

5. Consider the following iterative scheme:

$$x_{k+1} = (\alpha I - tA)x_k + tb$$

where $\alpha \geq 1$. Suppose that A is symmetric positive definite matrix in $\mathbb{R}^{n\times n}$, with eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0$.

(a) Show that the above scheme converges if and only if $\frac{\alpha-1}{\lambda_1} < t < \frac{\alpha+1}{\lambda_n}$.

(b) Prove that the optimal t, in the sense of rate of convergence, is $\frac{2\alpha}{\lambda_1+\lambda_n}$

(c) Suppose the scheme converges, show that the scheme converges to the solution for $Ax = b$ if $\alpha = 1$.

6. Consider an $n \times n$ matrix M given by:

$$M = \frac{1}{10}\begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ 1 & & 0 & \ddots & \\ \vdots & & & \ddots & -1 \\ 1 & & & & 0 \end{bmatrix}$$

Show the convergence of the following iterative scheme:

$$x_{k+1} = Mx_k + b$$

where $b \in \mathbb{R}^n$.