THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 3 Suggested Solution

1. Consider the following system of equations:

$$\begin{aligned} -3x + 3y - 6z &= 4 \\ -4x + 7y - 8z &= 8 \\ 5x + 7y - 9z &= 12 \end{aligned}$$

(a) Determine whether the Jacobi method converges.

(b) Using initial approximation $x^{(0)} = (1, 0, 0)^T$, conduct the first two Jacobi iterations.

Solution:

(a) We have

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -9 \end{pmatrix}; P = \begin{pmatrix} 0 & -3 & 6 \\ 4 & 0 & 8 \\ -5 & -7 & 0 \end{pmatrix}$$

so

$$D^{-1}P = \begin{pmatrix} 0 & 1 & -2 \\ \frac{4}{7} & 0 & \frac{8}{7} \\ \frac{5}{9} & \frac{7}{9} & 0 \end{pmatrix}$$

It can be verified that the spectral radius $\rho(D^{-1}P) \approx 0.8133 < 1$, so the Jacobi method converges.

(b) $x^{(1)} = D^{-1}Px^{(0)} + D^{-1}b = (\frac{-4}{3}, \frac{12}{7}, \frac{-7}{9})^T$;
$x^{(2)} = D^{-1}Px^{(1)} + D^{-1}b = (\frac{122}{63}, \frac{-32}{63}, \frac{-20}{27})^T$.

2. Consider the following system of equations:

$$\begin{aligned} -3x + 3y - 6z &= 4 \\ -4x + 7y - 8z &= 8 \\ 2x + 7y - 9z &= 12 \end{aligned}$$

(a) Determine whether the Gauss-Seidel method converges.

(b) Using initial approximation $x^{(0)} = (1, 1, 1)^T$, conduct the first two Gauss-Seidel iterations.

Solution:

(a) We have

$$L_* = \begin{pmatrix} -3 & 0 & 0 \\ -4 & 7 & 0 \\ 2 & 7 & -9 \end{pmatrix}; U = \begin{pmatrix} 0 & -3 & 6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

so

$$L_*^{-1}U = \begin{pmatrix} 0 & 1 & -2 \\ 0 & \frac{4}{7} & 0 \\ 0 & \frac{2}{3} & -\frac{4}{9} \end{pmatrix}$$

It can be verified that the spectral radius $\rho(L_*^{-1}U) = \frac{4}{7} < 1$, so the Gauss-Seidel method converges.

(b) $x^{(1)} = L_*^{-1}Ux^{(0)} + L_*^{-1}b = (-\frac{7}{3}, \frac{20}{21}, -\frac{10}{9})^T$;
$x^{(2)} = L_*^{-1}Ux^{(1)} + L_*^{-1}b = (\frac{116}{63}, \frac{136}{147}, -\frac{116}{567})^T$.

3. Consider the following system of equations:

$$\begin{aligned} -3x - 2y - z &= 1 \\ -4x + 4y - 6z &= 2 \\ -2x - 3y + 5z &= 3 \end{aligned}$$

(a) Determine whether the SOR method converges if $\omega = 1.2$.
(b) Determine whether the SOR method converges if $\omega = 1.4$.
(c) Using initial approximation $x^{(0)} = (0, 0, -1)^T$, conduct the first two SOR iterations where $\omega = 1.2$.

Solution:

We have

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}; L = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}; U = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) When $\omega = 1.2$, let

$$M = (D - \omega L)^{-1}((1-\omega)D + \omega U) = \begin{pmatrix} -\frac{1}{5} & -\frac{4}{5} & -\frac{2}{5} \\ -\frac{6}{25} & -\frac{29}{25} & \frac{33}{25} \\ -\frac{168}{625} & -\frac{762}{625} & \frac{349}{625} \end{pmatrix}$$

It can be verified that the spectral radius $\rho(M) \approx 0.8799 < 1$, so the SOR method converges.

(b) When $\omega = 1.4$, let

$$M' = (D - \omega L)^{-1}((1-\omega)D + \omega U) = \begin{pmatrix} -\frac{2}{5} & -\frac{14}{15} & -\frac{7}{15} \\ -\frac{14}{25} & -\frac{128}{75} & \frac{217}{150} \\ -\frac{434}{625} & -\frac{3668}{1875} & \frac{2077}{3750} \end{pmatrix}$$

It can be verified that the spectral radius $\rho(M') \approx 1.196 > 1$, so the SOR method diverges.

(c) According to (a), we have:
$x^{(1)} = Mx^{(0)} + \omega(D - \omega L)^{-1}b = (0, -\frac{6}{5}, \frac{7}{125})^T$;
$x^{(2)} = Mx^{(1)} + \omega(D - \omega L)^{-1}b = (\frac{336}{625}, \frac{4956}{3125}, \frac{164743}{78125})^T$.

4. Recall in Homework 2, we discussed an alternative definition for 2D DFT. Here, we introduce a more natural definition for 2D DFT. What 2D DFT does is actually applying DFT horizontally or vertically, and then apply DFT on the other direction.
Let $F \in \mathbb{C}^{N \times N}$. We define 2D DFT as

$$\hat{F}(m, n) = DFT(F)(m, n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k, l) e^{-2\pi i \frac{mk+nl}{N}}$$

(a) Recall 1D DFT is given by $\hat{f} = \frac{1}{N}\overline{A_\omega} f$ where $f \in \mathbb{C}^n$ is a column vector.
By applying DFT on each row of F, and second DFT on each column, show that the 2D DFT of F is can be written as

$$\hat{F} = \frac{1}{N^2}\overline{A_\omega} F \overline{A_\omega}$$

(b) Given the computation cost for 1D FFT is of $O(N\log(N))$. By applying FFT in above approach, we can get 2D FFT.
What is the computation cost for 2D FFT?

Solution:

(a) Applying first DFT, we have

$$\frac{1}{N}\overline{A_\omega}F^T$$

Applying second DFT, we have

$$\frac{1}{N}\overline{A_\omega}\left(\frac{1}{N}\overline{A_\omega}F^T\right)^T = \frac{1}{N^2}\overline{A_\omega}F\overline{A_\omega}$$

Write $\tilde{F} = \frac{1}{N^2}\overline{A_\omega}F\overline{A_\omega}$.
Next, we need to show that $\hat{F} = \tilde{F}$.
Note for $A, B \in \mathbb{C}^{N\times N}$,

$$(AB)(m,n) = \sum_{a=0}^{N-1} A(m,a)B(a,n)$$

Then

$$\begin{aligned}
\tilde{F}(m,n) &= \frac{1}{N^2}(\overline{A_\omega}F\overline{A_\omega})(m,n) \\
&= \frac{1}{N^2}\sum_{a=0}^{N-1}\overline{A_\omega}(m,a)(F\overline{A_\omega})(a,n) \\
&= \frac{1}{N^2}\sum_{a=0}^{N-1}\overline{A_\omega}(m,a)\sum_{b=0}^{N-1}F(a,b)\overline{A_\omega}(b,n) \\
&= \frac{1}{N^2}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1}F(a,b)\times\overline{\omega^{m\times a}}\times\overline{\omega^{b\times n}} \\
&= \frac{1}{N^2}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1}F(a,b)e^{-2\pi i\frac{ma+nb}{N}} \\
&= \hat{F}(m,n)
\end{aligned}$$

So, we have $\hat{F} = \tilde{F} = \frac{1}{N^2}\overline{A_\omega}F\overline{A_\omega}$

(b) In the first DFT, we need N FFT, which cost $O(N^2\log(N))$. And so as the second FFT. So, the overall computation cost is of $O(N^2\log(N))$.

5. Consider the following iterative scheme:

$$x_{k+1} = (\alpha I - tA)x_k + tb$$

where $\alpha \geq 1$. Suppose that A is symmetric positive definite matrix in $\mathbb{R}^{n\times n}$, with eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0$.

(a) Show that the above scheme converges if and only if $\frac{\alpha-1}{\lambda_1} < t < \frac{\alpha+1}{\lambda_n}$.

(b) Prove that the optimal t, in the sense of rate of convergence, is $\frac{2\alpha}{\lambda_1+\lambda_n}$

(c) Suppose the scheme converges, show that the scheme converges to the solution for $Ax = b$ if $\alpha = 1$.

Solution:

(a) Since A is symmetric positive definite, all eigenvalues of A are positive real numbers, and $\lambda_n = \rho(A)$. This implies that all eigenvalues of $B = \alpha I - tA$ are real as well. Note that (λ, v) is an eigen-pair of A iff $(\alpha - t\lambda, v)$ is an eigen-pair of B. Assume that $\frac{\alpha-1}{\lambda_1} < t < \frac{\alpha+1}{\lambda_n}$, then for all $i = 1, \cdots, n$, since $\alpha \geq 1$,

$$\alpha - 1 < \lambda_1 t \leq \lambda_i t \leq \lambda_n t < \alpha + 1$$

$$1 > \alpha - \lambda_i t > -1$$

This implies that $\rho(\alpha I - tA) < 1$, and implies convergence. Also, if $t \geq \frac{\alpha+1}{\lambda_n}$ or $t \leq \frac{\alpha-1}{\lambda_1}$, then $\rho(\alpha I - tA) \geq 1$.

(b) Obviously, $\rho(\alpha I - tA) = \max\{|\alpha - t\lambda_1|, |\alpha - t\lambda_n|\}$. Minimum of ρ is attained when

$$\begin{aligned} |\alpha - t\lambda_1| &= |\alpha - t\lambda_n| \\ \alpha - t\lambda_n &= t\lambda_1 - \alpha \\ \implies \quad t &= \frac{2\alpha}{\lambda_1 + \lambda_n} \end{aligned}$$

(c) Suppose $\alpha = 1$. Then we have:

$$(\alpha I - tA)x^* + tb = x^* - t(Ax^* - b) = x^*$$

which implies that the iterative scheme converges to x^*.

6. Consider an $n \times n$ matrix M given by:

$$M = \frac{1}{10}\begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ 1 & & 0 & \ddots & \\ \vdots & & & \ddots & -1 \\ 1 & & & & 0 \end{bmatrix}$$

Show the convergence of the following iterative scheme:

$$x_{k+1} = Mx_k + b$$

where $b \in \mathbb{R}^n$.

Solution:

By Gershgorin Circle Theorem, we can see that all eigenvalues are in the ball centred at 0 with radius 0.2. Then, we have the spectral radius is less than 1, and hence the scheme is convergent.