THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 9 solutions
10th November 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) $f(x) = x^2$ is continuous on $\mathbb{R}$. Given any $c \in \mathbb{R}$, and any $\epsilon > 0$, we pick $\delta = \min\{1, \frac{\epsilon}{2|c|+1}\} > 0$, then if x is in the range of $0 < |x - c| < \delta$, we have $|x| \leq |c| + |x - c| < |c| + \delta \leq |c| + 1$, and so $|x + c| \leq |x| + |c| \leq 2|c| + 1$.

$$|x^2 - c^2| = |x + c| \cdot |x - c| < (2|c| + 1)\delta \leq \epsilon.$$

(b) $f(x) = \frac{x}{x^2-1}$ is continuous on $\mathbb{R} \setminus \{\pm 1\}$. For any $c \in \mathbb{R} \setminus \{\pm 1\}$ and any $\epsilon > 0$, firstly consider $r = \min\{|c + 1|, |c - 1|\}/2$. Then for any $x \in (c - r, c + r)$, we have by triangle inequality $\min\{|x+1|, |x-1|\} \geq \min\{||c+1|-|c-x||, ||c-1|-|c-x||\}$. Note that both $|c + 1|, |c - 1|$ greater than or equal to $\min\{|c + 1|, |c - 1|\} = 2r > |c - x|$, so $\min\{||c + 1| - |c - x|, ||c - 1| - |c - x||\} = \min\{|c + 1| - |c - x|, |c - 1| - |c - x|\} = \min\{|c + 1|, |c - 1|\} - |c - x| > 2r - r = r$. Finally, from the above, for any x in the range of $0 < |x - c| < r$, we have

$$\frac{|xc| + 1}{|(x^2 - 1)(c^2 - 1)|} \leq \frac{|c|(|c| + r) + 1}{r^2 \cdot (2r)^2} = \frac{|c|^2 + |c|r + 1}{4r^4} =: K$$

Now given any $\epsilon > 0$, we may take $\delta = \min\{r, \epsilon/K\}$, then for x in the range of $0 < |x - c| < \delta$,

$$\begin{aligned}\left|\frac{x}{x^2 - 1} - \frac{c}{c^2 - 1}\right| &= \left|\frac{xc^2 - cx^2 + c - x}{(x^2 - 1)(c^2 - 1)}\right| \\ &\leq \frac{|xc| + 1}{|(x^2 - 1)(c^2 - 1)|}|x - c| \\ &< \frac{|c|^2 + |c|r + 1}{4r^4}\delta \\ &= K\delta \leq \epsilon.\end{aligned}$$

We would also like to show that f is discontinuous at ± 1. Simply consider the sequences $(x_n) = (\sqrt{1 + \frac{1}{n}}) \to 1$ and $(-x_n) \to -1$. We have $f(x_n) = n\sqrt{1 + \frac{1}{n}} = \sqrt{n^2 + n} \to \infty$ and $f(-x_n) = -\sqrt{n^2 + n} \to -\infty$. So by sequential criterion f cannot be continuous at those points.

(c) First, we claim that $f(x)$ is discontinuous for x non-zero rational number. This can be simply seen by sequential criterion. Given any rational $\frac{p}{q}$, by density of $\mathbb{R} \setminus \mathbb{Q}$, there is a sequence (r_n) of irrational number so that $\lim r_n = \frac{p}{q}$, then $\lim f(r_n) = \lim r_n = \frac{p}{q} \neq f(\frac{p}{q})$ if $\frac{p}{q} \neq 0$.

Next, we show that $f(x)$ is continuous at $x = 0$, given $\epsilon > 0$, simply take $\delta = \epsilon$, then for $0 < |x| < \delta$, if x is irrational, $|f(x)| = |x| < \epsilon$, and if $x = \frac{p}{q}$ is rational, $|f(\frac{p}{q})| = |p| \cdot |\sin(1/q)| \leq |p| \cdot |1/q| = |x| < \epsilon$. In the above, we have used the inequality $|\sin a| \leq |a|$.

Finally, we will prove that for c irrational, $f(x)$ is continuous at c. We will need the following fact, which we take for granted, $\lim_{q\to\infty} \frac{\sin 1/q}{1/q} = 1$. First, given any $\epsilon > 0$, we may pick $1 > a > 0$ and $\epsilon/2 > \delta' > 0$ so that $a(r + \delta') < \epsilon/2$. Given such a, by the limit we mentioned, there exists $N \in \mathbb{N}$ so that for $q \geq N$, we have $1 - a < q\sin(1/q) < 1 + a$. For this N, we note by a similar argument as in Thomae's function, there are finitely many rational numbers within distance at most 1 to c, whose reduced form has denominator q less than N. So there must exist $\delta'' > 0$ small enough so that any $\frac{p}{q} \in \mathbb{Q} \cap (c - \delta'', c + \delta'')$ written in reduced form has $q \geq N$. Now we take $\delta = \min\{\delta', \delta''\}$, then for $x \in (c - \delta, c + \delta)$, if x is irrational, then $|f(x) - c| = |x - c| = \delta < \epsilon/2 < \epsilon$. If $x = \frac{p}{q}$ is rational and written in reduced form, then

$$
\begin{aligned}
|f(x) - f(c)| &= \left| \frac{p}{q} \cdot q \sin\left(\frac{1}{q}\right) - c \right| \\
&\leq \max\left\{ \left| \frac{p}{q}(1 + a) - c \right|, \left| \frac{p}{q}(1 - a) - c \right| \right\} \\
&\leq \left| \frac{p}{q} - c \right| + a \cdot \frac{p}{q} \\
&< \delta + a(r + \delta) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

2. Yes, if $f + g$ was continuous, then along with f being continuous would imply $g = (f + g) - f$ is also continuous, which would be a contradiction.

3. Take $f(x) = \frac{1}{x}$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$, clearly $f(x)$ is discontinuous at $x = 0$. If we take $g = f$ also, then $g \circ f(x) = x$ for any $x \in \mathbb{R}$, which is a continuous function.

4. First, note that $f(0) = f(0 + 0) = f(0) + f(0)$ implies that $f(0) = 0$. Suppose that f is continuous at c, then given $\epsilon > 0$, we can find $\delta > 0$ so that $0 < |y - c| < \delta$ implies $|f(y) - f(c)| = |f(y - c)| < \epsilon$. Now if c' is any other point in $\mathbb{R}$, taking the same $\delta > 0$, note that if $0 < |x - c'| < \delta$, then $0 < |(x - c' + c) - c| < \delta$, i.e. $y = x - c' + c$ satisfies the premise above, so we have $\epsilon > |f(x - c' + c - c)| = |f(x) - f(c') + f(c) - f(c)| = |f(x) - f(c')|$.

5. First note that for all x, $g(x) = g(0 + x) = g(0)g(x)$. If $g(x) = 0$ for all x, then it is a constant function, and hence is continuous. Otherwise, $g(x) \neq 0$ for some x, then dividing through $g(x)$, we must have $g(0) = 1$. Furthermore, $g(x)$ is non-vanishing, if say $g(x) = 0$ for some x, then $1 = g(0) = g(x - x) = g(x)g(-x) = 0$, which is absurd. Now

for any $c \neq 0$, given any $\epsilon > 0$, by continuity of g at 0, we have $\delta > 0$ so that $0 < |x| < \delta$ implies $|g(x) - 1| < \epsilon/|g(c)|$. For the same δ, if x is in the range of $0 < |x - c| < \delta$, note that $|g(x - c) - 1| < \epsilon/|g(c)|$. Therefore $\epsilon > |g(c)g(x - c) - g(c)| = |g(x) - g(c)|$.

6. (a) We will prove that the complement D_ϵ^c is open. Given any $c \in D_\epsilon^c$, by assumption, there is some $\delta_x > 0$ so that for all $x, y \in (c - \delta_c, c + \delta_c)$, we have $|f(x) - f(y)| < \epsilon$. Suppose d is another point in $(c - \delta_c, c + \delta_c)$, then simply take $\delta_d = \min\{|c + \delta_c - d|, |c - \delta_c - d|\}$, we have $(d - \delta_d, d + \delta_d) \subset (c - \delta_c, c + \delta_c)$, and therefore for any $x, y \in (d - \delta_d, d + \delta_d)$, we have $|f(x) - f(y)| < \epsilon$, i.e. $(c - \delta_c, c + \delta_c) \subset D_\epsilon^c$. Thus we may write as an arbitrary union of open intervals

$$D_\epsilon^c = \bigcup_{c \in D_\epsilon^c} (c - \delta_c, c + \delta_c).$$

   (b) Suppose $\epsilon_1 < \epsilon_2$, if x is ϵ_1-continuous, then there is δ so that for any $y, z \in (x - \delta, x + \delta)$ we have $|f(y) - f(z)| < \epsilon_1 < \epsilon_2$, so x is automatically ϵ_2-continuous. By contrapositive, if x is not ϵ_2-continuous, then it is not ϵ_1-continuous, i.e. $D_{\epsilon_2} \subset D_{\epsilon_1}$.

   (c) If f is continuous at c, then for any $\epsilon > 0$, there is some δ so that whenever $0 < |x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon/2$. Then for any $x, y \in (c - \delta, c + \delta)$, we have $|f(x) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \epsilon/2 + \epsilon/2 = \epsilon$. So f is ϵ-continuous at c, for arbitrary ϵ. In our notation, $\bigcup_{\epsilon>0} D_\epsilon \subset D_f$.

   (d) If f is not continuous at c, then there exists some $\epsilon > 0$ so that for any $\delta > 0$, there is some x_δ with $0 < |x_\delta - c| < \delta$ so that $|f(x_\delta) - f(c)| \geq \epsilon$. In particular taking $x = x_\delta$ and $y = c$, we see that this implies that f is not ϵ-continuous at c. In terms of the subsets, this says that $D_f \subset \bigcup_{\epsilon>0} D_\epsilon$.

   Now we claim that $\bigcup_{\epsilon>0} D_\epsilon = \bigcup_n D_{\frac{1}{n}}$. The ($\supseteq$) direction is trivial, as we are taking union over a subfamily. For the ($\subseteq$) direction, simply note that by part (b), if $x \in D_\epsilon$, then by AP we may take some $n \in \mathbb{N}$ big enough so that $\frac{1}{n} < \epsilon$, then $D_\epsilon \subset D_{\frac{1}{n}}$. This concludes the proof.