THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 6
20th October 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) Given $\epsilon > 0$, pick $N \in \mathbb{N}$ so that $N > 8/\epsilon$, then for any $n, m \geq N$,

$$\begin{aligned} |x_n - x_m| &= \left| \frac{n^2 - 1}{n^2 + 3} - \frac{m^2 - 1}{m^2 + 3} \right| \\ &= \left| \frac{4}{n^2 + 3} - \frac{4}{m^2 + 3} \right| \\ &\leq \frac{4}{n^2 + 3} + \frac{4}{m^2 + 3} \\ &< \frac{4}{N} + \frac{4}{N} = \frac{8}{N} < \epsilon. \end{aligned}$$

   (b) Let $N \geq 1$ to be fixed later, for $n > m \geq N$, consider

$$\begin{aligned} |x_n - x_m| &= \left| \frac{(-1)^m}{(m+1)!} + \ldots + \frac{(-1)^{n+1}}{n!} \right| \\ &\leq \frac{1}{(m+1)!} + \ldots + \frac{1}{n!} \\ &< \frac{1}{2^m} + \ldots + \frac{1}{2^{n-1}} \\ &< \frac{1}{2^N} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^{N-1}} \leq \frac{1}{N}. \end{aligned}$$

   Given $\epsilon > 0$, simply fix $N \in \mathbb{N}$ so that $N \geq 1/\epsilon$, then the above calculation shows that $|x_n - x_m| < \frac{1}{N} \leq \epsilon$ for $n, m \geq N$.

   (c) Notice that $x_n - x_{n-1} = -\frac{1}{2}(x_{n-1} - x_{n-2})$, therefore $|x_n - x_{n-1}| \leq \frac{1}{2}|x_{n-1} - x_{n-2}|$ and we may apply the result of Q3 to conclude that it is Cauchy, or just follow the proof of Q3 as it is just the same.

2. The answer is no, the sequence may not be Cauchy. A counterexample is given by the harmonic series $x_n := \sum_{k=1}^{n} \frac{1}{n}$, then $|x_{n+1} - x_n| = \frac{1}{n+1} \to 0$, despite x_n not being a Cauchy sequence.

3. If $x_1 = x_2$, then by the inequality, the whole sequence is constant. Otherwise let $N \geq 1$ to be fixed later, for $n > m \geq N$

$$\begin{aligned} |x_n - x_m| &= \left|\sum_{k=m}^{n-1}(x_{k+1} - x_k)\right| \\ &\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \\ &\leq \sum_{k=m}^{n-1} C^{k-1}|x_2 - x_1| \\ &< |x_2 - x_1| \sum_{k=N}^{\infty} C^{k-1} = \frac{C^{N-1}|x_2 - x_1|}{1 - C} \end{aligned}$$

Recall that for $0 < C < 1$, we may rewrite $C = \frac{1}{1+r}$ for some $r > 0$, and hence obtain from Bernoulli's inequality the bound

$$C^{N-1} = \frac{1}{(1+r)^{N-1}} \leq \frac{1}{1 + r(N-1)}.$$

So given $\epsilon > 0$, we may pick $N \in \mathbb{N}$ so that $N \geq \frac{1}{r}(\frac{|x_2 - x_1|}{(1-C)\epsilon} - 1) + 1$, so that when $n, m \geq N$, we have $|x_n - x_m| < \frac{|x_2 - x_1|}{(1-C)(1+r(N-1))} \leq \epsilon$ by the above calculation.

4. Suppose that (x_n) is a monotone sequence, and (x_{n_k}) is a Cauchy subsequence. By replacing x_n with $-x_n$ if necessary, we may assume that x_n is monotonic increasing. Now given $\epsilon > 0$, by Cauchy-ness of the subsequence, there is some $K \in \mathbb{N}$ so that $x_{n_k} - x_{n_j} < \epsilon$ for any $k > j \geq K$. Now simply take $N = n_K \in \mathbb{N}$, then for $n > m \geq N$, we can find some k so that $n_k > n > m \geq n_K$, then we have $x_n - x_m \leq x_{n_k} - x_{n_K} < \epsilon$.

5. Suppose (x_n) is a bounded and monotone sequence, again assume that x_n is monotone increasing, otherwise simply replace x_n by $-x_n$. We will prove by contradiction. Suppose on the contrary that (x_n) is not Cauchy. Then there is some $\epsilon > 0$ so that for any $N \in \mathbb{N}$ there is some $n > m \geq N$ where $x_n - x_m \geq \epsilon$. By monotonicity, we might as well take $m = N$.

   Start by taking $n_1 = 1$, then we have $n_2 > n_1$ with $x_{n_2} - x_{n_1} \geq \epsilon$. For this n_2, we can find $n_3 > n_2$ so that $x_{n_3} - x_{n_2} \geq \epsilon$. Inductively, we have $n_k > n_{k-1}$ so that $x_{n_k} - x_{n_{k-1}} \geq \epsilon$.

   We will now show that (x_n) must be unbounded above. Take any $M > x_1$, by Archimedean property there is some $K \in \mathbb{N}$ so that $K\epsilon > M - x_1$. Then we have

$$x_{n_{K+1}} = x_1 + \sum_{k=1}^{K}(x_{n_{k+1}} - x_{n_k}) \geq x_1 + K\epsilon > M.$$

   This is a contradiction. So (x_n) must be Cauchy to begin with.

6. Let A be a subset of $\mathbb{R}$ that is bounded above. The goal is to show that $\sup A$ exists using only Cauchy criterion and AP. First pick $a_1 \in A$ and b_1 any upper bound of A, consider $\frac{1}{2}(a_1 + b_1)$, if it is an upper bound of A, then we set $a_2 = a_1$ and $b_2 = \frac{1}{2}(a_1 + b_1)$.

Otherwise if it is not an upper bound, then we set $a_2 = \frac{1}{2}(a_1 + b_1)$ and $b_2 = b_1$. Repeat this process to obtain a_n and b_n inductively, by considering whether $\frac{1}{2}(a_{n-1} + b_{n-1})$ is an upper bound of A.

Note that by construction (a_n) is monotonic increasing, (b_n) is monotonic decreasing. And all a_n are not upper bound, meanwhile all b_n are upper bounds of A. By results of Q5, we know that (a_n) and (b_n) are Cauchy sequences (note that we used AP in the proof of Q5). By Cauchy criterion, $\lim a_n$ and $\lim b_n$ exists. Furthermore we have $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^{n-1}}(b_1 - a_1)$, therefore $\lim a_n = \lim b_n =: x$.

The claim is that $x = \sup A$. First, it is an upper bound of A because b_n's are upper bounds, i.e. for any $a \in A$, $b_n \geq a$. Since taking limits preserves order, we have $x = \lim b_n \geq a$ for any a. Next, x is also the least upper bound. By convergence of a_n, for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that $x - a_N < \epsilon$. Rearranging this gives $x - \epsilon < a_N$, since a_N is not an upper bound, $x - \epsilon$ is not an upper bound as well.

Remark: Cauchy criterion gives an alternative way to understand completeness. Now we have two intimately related notions that are equivalent (up to AP). The advantage is that both can be generalized in different contexts. For example, axiom of completeness (defined via supremum) can be generalized to partially ordered sets, meanwhile the Cauchy condition can be generalized to metric spaces.