THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 5 solutions
13th October 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) It is not closed because 0 is a limit point but $0 \notin (0,1]$. For any $1 > \epsilon > 0$, clearly $(-\epsilon, \epsilon) \cap (0,1] = (0,\epsilon)$ is non-empty. Therefore $(0,1]$ cannot be compact. The family $\{U_n\}_{n=1}^{\infty}$ where $U_n = (\frac{1}{n}, 2)$ defines an open cover of $(0,1]$ since for any $1 > \epsilon > 0$, $2 > \epsilon > \frac{1}{n}$ for some n. Say if this cover admits a finite subcover $\{U_{n_1}, ..., U_{n_k}\}$, then taking $N = \max\{n_1, ..., n_k\}$, it is clear that

$$\bigcup_{i=1}^{k} U_{n_i} = U_N = (1/N, 2) \not\ni \frac{1}{N+1}.$$

This gives a contradiction. So the given open cover does not admit any subcover.

(b) $[0,\infty)$ is closed. It suffices to show that any $x \in (-\infty, 0)$ cannot be a limit point. This is clear because by taking $\epsilon = |x| > 0$, we have $(x-\epsilon, x+\epsilon) = (2x, 0)$ does not contain any element of $[0,\infty)$.

However, $[0,\infty)$ is still not a compact subset since it is not bounded. We may explicitly take the open cover $\{U_n\}_{n=1}^{\infty}$ where $U_n = (-1, n)$. Again, if $\{U_{n_1}, ..., U_{n_k}\}$ is a finite subcover, then by taking $N = \max\{n_1, ..., n_k\}$ we would obtain

$$\bigcup_{i=1}^{k} U_{n_i} = U_N = (-1, N) \not\ni N+1.$$

So we obtain a contradiction.

(c) 0 is a limit point of X that does not belong to X. Since for any $\epsilon > 0$ by Archimedean property there is some $\frac{1}{n} \in (-\epsilon, \epsilon)$. Therefore X cannot be compact. An explicity open cover can be taken as $U_n = (\frac{1}{n} - \frac{1}{2n}, \frac{1}{n} + \frac{1}{2n}) = (\frac{1}{2n}, \frac{3}{2n})$. Clearly $\frac{1}{n} \in U_n$ so it is a cover of X. It does not admit any finite subcover because a finite collection of subsets of the form U_n cannot contain the point $\frac{1}{2N}$ where N is the largest index appearing in the finite subfamily.

(d) $\mathbb{Q} \cap [0,1]$ is not closed because any irrational $r \in [0,1]$ is a limit point of $\mathbb{Q} \cap [0,1]$ by density of $\mathbb{Q}$, i.e. for any $\epsilon > 0$, we may find some $q \in \mathbb{Q}$ so that $q \in (r-\epsilon, r+\epsilon)$. Hence X cannot be compact.

One may find a cover of $\mathbb{Q} \cap [0, 1]$ by picking a monontonic increasing sequence of $q_i \in \mathbb{Q} \cap [0, 1]$ that converges to some chosen irrational $r \in [0, 1]$. Now, we pick an irrational $r_1 < 0$, and irrationals $r_i \in (q_{i-1}, q_i)$. If we set $U_i = (r_i, r_{i+1})$, then $q_i \in U_i$ and the collections $\{U_i\}$ is disjoint. Also notice that $\{U_i\}$ is an open cover of $\mathbb{Q} \cap [0, r]$. Therefore we obtain an open cover of $\mathbb{Q} \cap [0, 1]$ by adding in a single open set $U_0 = (r, 2)$. Then we claim that $\{U_i\}_{i=0}^{\infty}$ is an open cover of X without any finite subcover. This is clear by construction because if we miss U_0 in the collection, then 1 is not in the union. And if we miss any U_n for $n > 0$, then q_n is not in the union. In fact, the cover we have constructed does not contain any proper subcover.

2. Method 1: We will use the bounded and closed definition of compactness. By boundedness, $\sup X$ exists. And for any $\epsilon > 0$, there is some $x > \sup X - \epsilon$, i.e. $X \cap (\sup X - \epsilon, \sup X + \epsilon) \neq \emptyset$. So $\sup X$ is a limit point, and by closedness we have $\sup X \in X$.

   Method 2: We will use the open cover definition of compactness. Suppose that $\sup X \notin X$, then consider the open cover $\{U_n\}$ where $U_n = (-\infty, \sup X - \frac{1}{n})$. It is an open cover of X since clearly $X \in \bigcup_n U_n = (-\infty, \sup X)$. However, it does not contain any finite subcover because if $\{U_{n_1}, ..., U_{n_k}\}$ is a subfamily, then by taking $N = \max\{n_1, ..., n_k\}$ we have $\bigcup_{i=1}^{k} U_{n_i} = U_N = (-\infty, \sup X - \frac{1}{N})$ but by definition of supremum there is some $x \in X$ so that $x > \sup X - \frac{1}{N}$, hence the said family cannot be a subcover.

   Really the two methods are the same...

3. Method 1: Using the bounded and closed definition of compactness, if K is a compact subset, and $A \subset K$ is a closed subset, then clearly $A \subset K$ is bounded. Therefore A will be compact.

   Method 2: Using the open cover definition. Suppose that A is a closed subset of compact K, let $\{U_i\}_{i \in I}$ be an open cover of A, then $\{U_i\}_{i \in I} \cup A^c$ is an open cover of K (recall that A is closed implies that A^c is open). By compactness of K there is a finite subcover $\{U_{i_1}, ..., U_{i_k}\}$ (possibly containing A^c as well). In any case, $\{U_{i_1}, ..., U_{i_k}\}$ would then be a finite subcover of A of the original cover, since $A^c \cap A = \emptyset$.

4. Let A, B be compact sets, suppose that $\{U_i\}_{i \in I}$ is an open cover of $A \cup B$, in particular it is an open cover of A and B individually. Therefore by compactness there are subcovers of A and B respectively, taking their union would produce a subcover that covers $A \cup B$.

   As for intersection, $A \cap B$ is a closed subset because A and B are closed, therefore by results of Q3, $A \cap B$ is compact.

   If you want a proof of this solely from the open cover definition, it is trickier because the statement does not hold for arbitrary (topological) spaces. It hinges on another crucial topological property of $\mathbb{R}$ known as being Hausdorff. The problem is that in general a compact set may not even be closed if we are considering complicated examples! So in any case, we have to make use of some results along the line that "compact sets are closed".

5. Method 1: Suppose that $\{K_n\}$ is a decreasing sequence of nonempty compact subsets. By boundedness $x_n := \sup K_n$ exists for all n, and since $K_{n+1} \subset K_n$, we have $x_{n+1} \leq x_n$. If y is a lower bound of K_1, it is a lower bound of all K_n, therefore we have a monotonic decreasing sequence $\{x_n\}$ that is also bounded below. By monotone convergence theorem $x = \lim x_n$ exists. Furthermore, by compactness and result of Q2 $x_n = \sup K_n \in K_n$. In

fact, for all $n \in N$, for $i \geq n$, we have $x_i \in K_n$. Therefore, the sequence is eventually lying entirely inside each of K_n. But each K_n is closed, so the limit $x \in K_n$ for arbitrary n, therefore x is in the intersection.

Method 2: Suppose for the sake of contradiction that $\bigcap_{n=1}^{\infty} = \emptyset$, then $\{U_n\}_{n=1}^{\infty}$ where $U_n = \mathbb{R} \setminus K_n$ is an open cover of K_1 since $\bigcup_{n=1}^{\infty} \mathbb{R} \setminus K_n = \mathbb{R} \setminus \bigcap_{n=1}^{\infty} K_n = \mathbb{R}$. By compactness of K_1 there is some finite subcover $\{U_{n_1}, ..., U_{n_k}\}$, again the union of open sets from this subcover can be seen to be $\mathbb{R} \setminus K_N$ where $N = \max\{n_1, ..., n_k\}$. So we have $K_1 \subset \mathbb{R} \setminus K_N$, however $K_N \subset K_1 \subset K_N^c$ implies that $K_N = \emptyset$, this is a contradiction.

(Q1d revisited). Now let's see another way of showing that $X = \mathbb{Q} \cap [0, 1]$ is non-compact. Suppose that it is compact, then fix an irrational $r \in [0, 1]$, we know by Q3 that $U_n = X \cap [r - \frac{1}{n}, r + \frac{1}{n}]$ are again compact since they are closed subsets of compact set. By density of $\mathbb{Q}$, all these U_n are nonempty, and so we have a nested sequence of compact sets. By Q5, $\bigcap_{n=1}^{\infty} U_n$ must be nonempty, in particular the limit of $x_n = \sup U_n = r + \frac{1}{n}$, which is equal to r, is an element of the intersection, so it must be in $Q \cap [0, 1]$. This is a contradiction since we assumed r to be irrational.