THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 4
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- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

Topology on $\mathbb{R}$

Motivated by Bolzano-Weierstrass theorem, we are interested to study limit of subsets, and those subsets which contain its limits. In view of this, we make the following definition.

Definition. Let $A \subset \mathbb{R}$, $x \in \mathbb{R}$ is called a limit point of A if for any $\epsilon > 0$, there exists some $a \in A$ so that $0 < |x - a| < \epsilon$. The collection of limit points of A is denoted by $D(A)$.

Definition. A subset $A \subset \mathbb{R}$ is called closed if $D(A) \subset A$.

Do note that an isolated point $\{x\}$ has no limit point because in the definition we do not allow a to be taken as x. However, this is merely a convention and does not affect the structure of closed subsets as we defined it.

We observe the following properties.

Proposition. (Properties of closed subsets)

a. Let $\{A_i\}_{i \in I}$ be an arbitrary family of closed subsets, then $\bigcap_{i \in I} A_i$ is again closed.

b. Let $A_1, ..., A_n$ be finitely many closed subsets, then $\bigcup_{i=1}^n A_i$ is again closed.

c. $\mathbb{R}$ and $\emptyset$ are closed.

d. A is closed if and only if the complement $A^c = \mathbb{R} \setminus A$ is a union of open intervals.

Proof.

a. Suppose that $x \in D(\bigcap_{i \in I} A_i)$, then for any $\epsilon > 0$, we have $a \in \bigcap_{i \in I} A_i$ so that $0 < |x - a| < \epsilon$. In particular, for any $i \in I$, $a \in A_i$, therefore by definition $x \in D(A_i)$ for all $i \in I$. Since A_i are closed, $D(A_i) \subset A_i$. So $x \in \bigcap_{i \in I} A_i$.

b. Suppose that $x \in D(\bigcup_{i=1}^n A_i)$, we will prove that $x \in D(A_i)$ for some i. Assume otherwise that $x \notin D(A_i)$ for all $1 \leq i \leq n$. Then for each i there is an $\epsilon_i > 0$ so that $(x - \epsilon_i, x + \epsilon_i) \cap A_i \setminus \{x\} = \emptyset$. Therefore, if we take $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\} > 0$, then $(x - \epsilon, x + \epsilon) \cap (\bigcup_{i=1}^n A_i) \setminus \{x\} \subset \bigcap_{i=1}^n ((x - \epsilon_i, x + \epsilon_i) \cap A_i) \setminus \{x\} = \bigcap \emptyset = \emptyset$. This implies that $x \notin D(\bigcup_{i=1}^n A_i)$, which is a contradiction.

   Now since $x \in D(A_i)$ for some i, and A_i is closed. We have that $x \in D(A_i) \subset A_i$. In particular, $x \in \bigcup_{i=1}^n A_i$.

c. Clearly we have $D(\mathbb{R}) = \mathbb{R}$ and $D(\emptyset) = \emptyset$.

d. ($\Rightarrow$) We will show that for each $x \in A^c$, there is some $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon) \subset A^c$. To see this, we prove by contradiction. Suppose otherwise that for all $\epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap A \neq \emptyset$. Since $x \notin A$, by closure of A, there is some $a \in (x-\epsilon, x+\epsilon) \cap A$ so that $x \neq a$. This precisely means that $x \in D(A)$. But A is closed, so $x \in D(A) \subset A$. This is a contradiction.

   Now given the above fact, we can write $A^c = \bigcup_{x \in A^c}(x-\epsilon_x, x+\epsilon_x)$. The RHS is a union of open intervals.

   ($\Leftarrow$) Suppose that the complement is a union of open intervals, i.e. $A^c = \bigcup_{i\in I}(a_i, b_i)$, where a_i, b_i could be $\pm\infty$. Then $A = \mathbb{R} \setminus \bigcup_{i\in I}(a_i, b_i) = \bigcap_{i\in I}(\mathbb{R} \setminus (a_i, b_i))$. It suffices to show that subsets of the form $\mathbb{R} \setminus (a, b)$ are closed, then by part (a), we can conclude that A is closed.

   If $a = -\infty$ and $b = \infty$, then $(a, b) = \mathbb{R}$, and we have noted already that $\emptyset$ is closed. If $a = -\infty$, $b \neq \infty$, then $\mathbb{R} \setminus (-\infty, b) = [b, \infty)$. To show that $D([b,\infty)) \subset [b, \infty)$, it suffices to show that any $x < b$ is not a limit point. This is clear because if we take $\epsilon = b - x > 0$, then the interval $(x-\epsilon, x+\epsilon) \cap [b, \infty) = (2x - b, b) \cap [b, \infty) = \emptyset$. The remaining cases are similar. ■

Remark. Note that arbitrary union of closed subsets may not be closed. For example, take $A_n = \{\frac{1}{n}\}$ the singleton. Then $D(A_n) = \emptyset$, but $D(\bigcup_{n=1}^{\infty} A_n) = \{0\} \not\subset \bigcup_{i=1}^{\infty} A_n$. The step where we took $\min$ of ϵ_i broke down if we want to take infinite union. Since we have to replace $\min$ by $\inf$, but the infimum can be 0, in that case we cannot obtain a contradiction.

The above observations motivate the general definition of a topological space. In many contexts, we are interested to consider spaces that are more general than $\mathbb{R}$, where one cannot always invoke on ϵ arguments, for examples for spaces that do not admit a distance function like the absolute value we consider on $\mathbb{R}$. Nevertheless, it is still desirable to talk about sequences, limits and the notions of nearby points.

Definition. A topological space consists of a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T} \subset P(X)$ is a collection of subsets of X called open sets, such that $\mathcal{T}$ satisfies the following axioms.

1. $X, \emptyset \in \mathcal{T}$.
2. Let $\{A_i\}_{i\in I}$ be an arbitrary family of elements in $\mathcal{T}$, then $\bigcup_{i\in I} A_i \in \mathcal{T}$.
3. Let $A_1, ..., A_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$.

Given the above, closed sets are defined as the complement of open subsets. Note that a subset can be open and closed at the same time. In the example of $\mathbb{R}$, open sets are simply the unions of open intervals.

We close this discussion by mentioning a certain interesting closed subset of $\mathbb{R}$. We highlight the effectiveness of studying equivalent properties of mathematical notions, where it is sometimes more convenient to prove results using another equivalent definition.

Definition. (Cantor set) Let $F_1 = [0, 1]$, define F_{n+1} inductively by deleting the middle one-third open interval from each interval of F_n. More specifically, $F_2 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup$

$[\frac{2}{3}, 1]$, $F_3 = [0, \frac{1}{3}] \setminus (\frac{1}{9}, \frac{2}{9}) \cup [\frac{2}{3}, 1] \setminus (\frac{7}{9}, \frac{8}{9})$, and so on. Alternatively, one can define explicitly $F_{n+1} = \frac{1}{3}F_n \cup (\frac{2}{3} + \frac{1}{3}F_n)$. The Cantor set $\mathcal{C}$ is defined as the intersection $\mathcal{C} = \bigcap_{n=1}^{\infty} F_n$.

The process of inductively subdividing $[0, 1]$ into three equal parts is equivalent to expanding a number into base 3 number system. More specifically, any number in $[0, \frac{1}{3})$ when expanded in base 3 is always of the form $0.0a_2a_3...$; numbers in $(\frac{1}{3}, \frac{2}{3})$ when expanded is always of the form $0.1a_2a_3...$; numbers in $(\frac{2}{3}, 1)$ is always of the form $0.2a_2a_3....$ Further subdividing the intervals into 3 equal parts corresponds to looking at one further digit in the expansion. Therefore, since the Cantor set is obtained from removing middle one-third interval, at each step we are removing numbers whose n-th digit in its base 3 expansion is not equal to 1. Notice however, there is a subtle point, where in the expansion, we have to replace terminating fractions by a repeating one, i.e. $\frac{1}{3} = 0.1_3 = 0.0222..._3$. In this case, $\frac{1}{3} \in \mathcal{C}$. From this description,we note that $\mathcal{C}$ is in fact uncountable, because at each digit, we can choose 0 or 2, so $|\mathcal{C}| = |2^{\mathbb{N}}|$, i.e. $\mathcal{C}$ is not countable.

One nice application of our discussion on closed subsets is that one can immediately see that $\mathcal{C}$ is closed. This is because it admits a description as the intersection of finite union of closed intervals. We know that closed intervals are closed, and so are their finite unions, and so are their arbitrary intersection. On the other hand, directly proving that $\mathcal{C}$ is closed from definition using limit point is trickier and one has to deal with the issue of certain numbers admitting more than one description in base 3 expansion.