# THE CHINESE UNIVERSITY OF HONG KONG
## Department of Mathematics
## MATH 2058 Honours Mathematical Analysis I 2022-23
## Tutorial 2 solutions
## 22nd September 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) Let $x_n = n/100$, it suffices to show that it is unbounded. For any $M > 0$, by Archimedean property there is some $n > 100M$, therefore $x_n > M$. Unbounded sequence is divergent by proposition 2.7.

   (b) We will show that $\lim x_n = 1$. To prove this, pick any $\epsilon > 0$, if $\epsilon > 1$, then for any $n \in \mathbb{N}$,
$$\left|\frac{n^2-1}{n^2+1} - 1\right| = \frac{2}{n^2+1} \leq 1 < \epsilon.$$
Otherwise $\epsilon < 1$, then by Archimedean property there exists some $N \in \mathbb{N}$ so that $N > \sqrt{\frac{2}{\epsilon} - 1}$, which is equivalent to saying $\epsilon > \frac{2}{N^2+1}$, then for any $n \geq N$
$$\left|\frac{n^2-1}{n^2+1} - 1\right| = \frac{2}{n^2+1} \leq \frac{2}{N^2+1} < \epsilon.$$

   (c) We will show that $\lim x_n = 0$. First note that for $n \geq 2$,
$$0 \leq \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n} + \sqrt{n} - 1} \leq \frac{1}{2\sqrt{n-1}}$$
Consider now the sequence $\{y_n\}$ where $y_1 = 1$ and $y_n = \frac{1}{2\sqrt{n-1}}$ for $n \geq 2$. Then by squeeze theorem (proposition 2.10), it suffices to show that $\lim y_n = 0$.
Now pick any $\epsilon > 0$, by Archimedean property there is some $N \in \mathbb{N}$ so that $N > \frac{1}{4\epsilon^2} + 1$, which is equivalent to $\epsilon > \frac{1}{2\sqrt{N-1}}$. Then for any $n \geq N$, we have
$$\left|\frac{1}{2\sqrt{n-1}}\right| \leq \frac{1}{2\sqrt{N-1}} < \epsilon.$$

   (d) Regardless of what the value of $a$ is, since $-1 \leq \cos(na) \leq 1$, the sequence $x_n = \cos(na)/n$ satisfies the following bound,
$$-\frac{1}{n} \leq \frac{\cos(na)}{n} \leq \frac{1}{n}.$$
By Monotone convergence theorem, $\lim \frac{1}{n} = \inf\{\frac{1}{n}\} = 0$ and $\lim(-\frac{1}{n}) = \sup\{-\frac{1}{n}\} = 0$. So by squeeze theorem, $\lim x_n = 0$.

(e) We will show that $\lim x_n = 0$. First note that

$$0 \leq \frac{n^2}{n^3+1} \leq \frac{n^2}{n^3} = \frac{1}{n}.$$

Once again the result follows from squeeze theorem.

(f) We will show that $x_n = \sqrt{n}$ is unbounded above, hence it is divergent. For any $M > 0$, by Archimedean property there is some $N \in \mathbb{N}$ so that $N > M$, so $x_{N^2} = \sqrt{N^2} = N > M$. So $x_n$ is unbounded.

2. Notice that $\lim |x_n| = 0$ is equivalent to saying that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that for $n \geq N$, $||x_n| - 0| = |x_n| < \epsilon$. The condition is identical to $\lim x_n = 0$.

3. Suppose that $c = 0$, then $cx_n = 0$ is just the constant sequence, which converges to $0 = cL$. We include the $\epsilon$-argument here for completeness: for any $\epsilon > 0$, take $N = 1$, then for $n \geq N = 1$, $|x_n - 0| = 0 < \epsilon$. Now suppose $c \neq 0$, then fix any $\epsilon > 0$, by assumption, with respect to $\epsilon' = \epsilon/|c| > 0$, we may find $N \in \mathbb{N}$ so that for $n \geq N$, we get $|x_n - L| < \epsilon'$. Then for the same $N$, we have $n \geq N$ gives $|cx_n - cL| = |c| \cdot |x_n - L| < |c|\epsilon' = \epsilon$ and we are done.

4. Suppose $\lim x_n = L$, this implies that given any $\epsilon > 0$, we can find $N \in \mathbb{N}$, so that for $n \geq N$, $|x_n - L| < \epsilon$. In fact, we may pick $N = 2k$ to be even because if $N$ is odd, then replacing $N$ by $N+1$ will give an even number. Then for the same $k \in \mathbb{N}$, if $m \geq k$, we have $|y_m - L| = |x_{2m} - L| < \epsilon$. The last inequality holds because $2m > 2k = N$.

5. (a) We can just compute

$$x_{n+1} - x_n = \frac{x_n^2 - 4x_n + 4}{4} = \frac{(x_n-2)^2}{4} \geq 0,$$

therefore $x_n$ is an increasing sequence.

(b) The base case $x_1 = 1 \leq 2$ is satisfies. Suppose now we know that $x_k \leq 2$, then $x_{k+1} = \frac{x_k^2+4}{4} \leq \frac{2^2+4}{4} = 2$. So inductively, we have $x_n \leq 2$ for any $n$.

(c) From the above, we know $\lim x_n = L$ exists according to monotone convergence theorem. Now the limit of the sequence $\{x_{n+1}\}$ is the same as the $\{x_n\}$, which can be readily seen by definition. Therefore, along with the fact that limit is compatible with arithmetic, the recurrence relation implies that $\lim x_{n+1} = ((\lim x_n)^2 + 4)/4$. So $L$ satisfies the relation $L = (L^2+4)/4$. Solving this yields $L = 2$.

6. (a) For this question, we will be using the Bernoulli's inequality. For $x > 0$, $n \in \mathbb{N}$, we have $(1+x)^n \geq 1 + nx$, which can be shown by simply doing binomial expansion and truncating the higher order terms. Suppose that $0 < a < 1$, then $a^{-1} > 1$ and therefore we can rewrite $x_n = a^n = \frac{1}{(1+c)^n}$ where $c > 0$. Then

$$0 \leq x_n = \frac{1}{(1+c)^n} \leq \frac{1}{1+nc} < \frac{1}{cn}.$$

By squeeze theorem, it suffices to show that $\lim 1/(cn) = 0$. This is clear because $\lim 1/n = 0$.

(b) $x_n = 1^n = 1$ is a constant sequence, hence it converges to $1$.

(c) For $a > 1$, we may write $x_n = a^n = (1+b)^n \geq 1+nb$, then $x_n$ can be easily seen to be unbounded since $1 + nb$ is. For any $M > 0$, by Archimedean principle, we may find $k \in \mathbb{N}$ large enough so that $kb > 1$ and $N \in \mathbb{N}$ so that $N > M$, then taking $n = Nk$, we get $x_{Nk} \geq 1 + (Nk)b > M$ as desired.

7. (a) We will use the monotone convergence theorem to show that limit of $x_n$ exists. First, we note that $x_n$ is monotonic increasing:

$$x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \geq \frac{2}{2n+2} - \frac{1}{n+1} = 0.$$

Then, we also have $x_n \leq 1$ simply by considering

$$x_n = \frac{1}{n+1} + ... + \frac{1}{2n+1} < \underbrace{\frac{1}{n} + ... + \frac{1}{n}}_{n \text{ terms}} = 1.$$

Therefore limit of $x_n$ exists.

(b) The argument is wrong because $x_n$ is not a sum of $n$ sequences, the number of terms in the sum depends on $n$.

Remark: The limit of $x_n$ turns out to be $\ln 2$, this can be shown by noting that $x_n$ is actually the same as Riemann sum of the function $f(x) = \frac{1}{x}$ from 1 to 2. Therefore, the limit approaches $\int_1^2 \frac{1}{x} dx = \ln 2$.

8. Suppose that $x_n$ is an integer sequence converging to some limit $L$, then pick $\epsilon = \frac{1}{2}$, we have $|x_n - L| < \frac{1}{2}$ for $n \geq N$ for some $N \in \mathbb{N}$, this in particular means that $x_n$ assumes the same value for $n \geq N$. Otherwise, if $x_m \neq x_n$ for some pair of $n, m \geq N$, then $|x_m - x_n| \geq 1$. But by triangle inequality $|x_n - x_m| \leq |x_n - L| + |x_m - L| < 1$. This is a contradiction.

9. Suppose $\lim x_n = 0$, then for any $\epsilon > 0$, we can find some $N \in \mathbb{N}$ so that if $n \geq N$, we have $x_n \leq \epsilon^2$ for $n \geq N$, this in particular implies $\sqrt{x}_n \leq \epsilon$ for $n \geq N$. So we are done.

10. For each $n \in \mathbb{N}$, by density of $S$, there exists some element $x_n \in S \cap (r - \frac{1}{n}, r + \frac{1}{n})$. We obtain a sequence $\{x_n\}$ inside $S$. We will now prove that $\lim x_n = r$. Pick any $\epsilon > 0$, then by Archimedean property we have $\frac{1}{N} < \epsilon$ for some $N \in \mathbb{N}$. Then for any $n \geq N$, we have $|x_n - r| < \frac{1}{n} \leq \frac{1}{N}\epsilon$ by construction of $x_n$. Hence the result.