THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 1 solutions
15th September 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) First note that for any $q \in \mathbb{Q}$, $q^2 \geq 0$, so X is bounded below. By completeness axiom, an infimum must exist. To see that 0 is precisely the infimum, consider any $\epsilon > 0$, without loss of generality, we may assume that $1 > \epsilon$, otherwise replace ϵ with a smaller ϵ'. By Archimedean property, specifically corollary 1.9, we can find $n \in \mathbb{N}$ so that $q = \frac{1}{n} \in \mathbb{Q}$ satisfies $1 > \epsilon > q > q^2$. Hence $0 + \epsilon$ is no longer a lower bound, and 0 is indeed the infimum. As for supremum, $\mathbb{N} \subset X$ and $\mathbb{N}$ is unbounded, if $\sup X$ exists that would imply $\sup \mathbb{N}$ exists, which is a contradiction.

   (b) Note that $|(-1)^n/n| \leq 1/n \leq 1$, so X is bounded and has both supremum and infimum. Consider when $n = 2k$ is even, it is clear that $(-1)^{2k} 1/(2k) = 1/(2k) > 1/(2k+2)$. Therefore $1/2$ is in fact the maximum of X. Similarly, for $n = 2k+1$ is odd, $(-1)^{2k+1} 1/(2k+1) = -1/(2k+1) < -1/(2k+3)$ and so -1 is the minimum of X.

   (c) We may rearrange the condition $4x - x^2 > 3$ as $1 > x^2 - 4x + 4 = (x-2)^2$. Thus the condition is equivalent to that $1 > |x-2|$, in other words $x \in (1, 3)$. The supremum and infimum of a bounded interval are just the two endpoints respectively. To see this, clearly 3 is an upper bound of $(1, 3)$. And for any $\epsilon > 0$, we may find $n \in \mathbb{N}$ so that $3 - \epsilon < 3 - \frac{1}{n} \in (1, 3)$. The argument is similar for showing that 1 is the infimum.

   (d) For any $r \in \mathbb{R}$ and $q \in \mathbb{Q}$, $|q - r| \geq 0$ so 0 is a lower bound. By proposition 1.12, for any $\epsilon > 0$, there exists a rational number q in the interval $(r - \epsilon, r + \epsilon)$, therefore such q satisfies $|q - r| < \epsilon$. Hence 0 is the infimum. As for supremum, it is clear that X is unbounded above.

2. Suppose that $\sup(A)$ exists, then A is bounded above, say there is $x \in \mathbb{R}$ so that $x > a$ for any $a \in A$, therefore $-x < -a$ for any $a \in A$. This is equivalent to saying that $-A$ is bounded below, so by completeness admits an infimum. To prove the equality $\inf(-A) = -\sup(A)$, we will use the ϵ-characterization. First, take $x = \sup(A)$, we note that $-x$ is a lower bound of $-A$. Suppose not, then there is an element $-a \in -A$ so that $-x > -a$, this implies $x < a$ for $a \in A$. This contradicts the fact that x is an upper bound of A. Now, to see that $-x$ is the infimum, consider $-x + \epsilon$ for some $\epsilon > 0$, since x itself is the supremum, we can find $a \in A$ so that $x - \epsilon < a$, flipping the sign yields $-x + \epsilon > -a \in -A$. This completes the proof.

3. As subsets, we have $A - B = A + (-B)$. So by proposition 1.6 and Q2, we have

$$\inf(A - B) = \inf(A + (-B)) = \inf(A) + \inf(-B) = \inf(A) - \sup(B).$$

4. To prove the claim, it suffices to show that for any $\epsilon > 0$, we have $\sup(A) < \inf(B) + \epsilon$. Fix any $\epsilon > 0$, by property of infimum, there exists $b \in B$ so that $a < b < \inf(B) + \epsilon$ holds for any $a \in A$. Since b is an upper bound of A by assumption, $\sup(A) \leq b$ and this completes the proof.

5. Suppose x be the infimum of T, in particular it is a lower bound of $S \subset T$, therefore $x = \inf T \leq \inf S$. The inequality for supremum is similar.

6. If S contains an upper bound of itself, say s_0, then $s_0 \geq s$ for any $s \in S$. In particular if we pick any $\epsilon > 0$, clearly $s_0 - \epsilon < s_0 \in S$. So $s_0 - \epsilon$ is no longer an upper bound, so s_0 is the supremum. Uniqueness is clear since $s_0 \geq s_1$ and $s_1 \geq s_0$ implies $s_0 = s_1$.

7. Suppose we know $\inf\{1/n : n \in \mathbb{N}\} = 0$, by ϵ-characterization, this means that for any $\epsilon > 0$, we can find an element $1/n$ so that $1/n < \epsilon$. To prove the Archimedean property, let $M \in \mathbb{R}$, if $M \leq 0$ then we are done since $M \leq 0 < 1$. Otherwise M is positive, and taking $1/M = \epsilon > 0$, we can find $1/n < \epsilon = 1/M$. Therefore such integer n satisfies $n > M$.

8. (a) Again, we will show instead that for any fixed $\epsilon > 0$, we have $\sup_{x\in D} f(x) - \epsilon < \sup_{x\in D} g(x)$. By ϵ-characterization, there is some $x_0 \in D$ so that $\sup_{x\in D} f(x) - \epsilon < f(x_0)$. Together with $f(x_0) \leq g(x_0) \leq \sup_{x\in D} g(x)$, we are done.
   (b) This follows immediately from Q4, by taking $f(D) = A$ and $g(D) = B$.
   (c) Let $D = [0, 2]$, $f(x) = x$ and $g(x) = x + 1$. Then clearly $f(x) \leq g(x)$ for any x. But $\sup_{x\in[0,2]} f(x) = 2$, meanwhile $\inf_{x\in[0,2]} g(x) = 1$.

9. (The original problem contains a typo that has since been fixed, please see the corrected version.)

   (a) We will show that $f(x) = \sup_{y\in D}(2x + y) = 2x + 1$ for any fixed $x \in D$. The argument is essentially the same as showing that 1 is the supremum of the interval $(0, 1)$. One can see that for fixed $x_0 \in D$, the image of $h(x_0, y) = 2x_0 + y$ is given by the interval $(2x_0, 2x_0 + 1)$ which has supremum $2x_0 + 1$. By the same argument, we can show that $\inf_{x\in D} f(x) = 1$ since $f(D)$ is just the interval $(1, 3)$.
   (b) The argument is identical to the previous part, $g(y)$ is equal to y since the image of h for a fixed y is given by the interval $(y, y + 2)$. Then $\sup_{y\in D} g(y) = \sup(0, 1) = 1$. Note that the answer coincides to that of part (a).

   (a) For fixed $x \in D = (0, 1)$, clearly we can find some $y \in (0, 1)$ so that $y \leq x$, like just take $y = x$. Then $h(x, y) = 1$ must be the supremum since the function h can only take two values. So $f(x) = \sup_{y\in D} h(x, y) = 1$ regardless of what x is, so $\inf_{x\in D} f(x) = 1$.
   (b) Similarly for fixed $y \in (0, 1)$, we can find $x \in (0, 1)$ so that $x < y$, for such pair we get $h(x, y) = 0$, hence $g(y) = \inf_{x\in D} h(x, y) = 0$ is again a constant. So $\sup_{y\in D} g(y) = 0$. Note that the answer differs from that of part (a).

10. Again, we will prove the inequality by equivalently showing that $\sup_{y\in Y}\inf_{x\in X} h(x,y)-\epsilon < \inf_{x\in X}\sup_{y\in Y} h(x,y)$ for any $\epsilon > 0$. Consider the LHS of the inequality, by ϵ-characterization, we can find some $y_0 \in Y$ so that

$$\sup_{y\in Y}\inf_{x\in X} h(x,y) - \epsilon < \inf_{x\in X} h(x,y_0).$$

Now clearly for any $x \in X$, by definition of supremum,

$$h(x,y_0) \leq \sup_{y_0\in Y} h(x,y_0).$$

Applying the infimum version of Q8a, we have

$$\inf_{x\in X} h(x,y_0) \leq \inf_{x\in X}\sup_{y_0\in Y} h(x,y_0).$$

Combined with the inequality above, we get the desired result.