THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 11 solutions
1st December 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Since the two limits exist, given any $\epsilon > 0$, we may find $M > 0$ so that if $x \geq M$, we have $|f(x) - L| < \epsilon/2$ and if $x \leq -M$, we have $|f(x) - \ell| < \epsilon/2$. Then for arbitrary $x, y \geq M$. $|f(x) - f(y)| \leq |f(x) - L| + |f(y) - L| < \epsilon$, and likewise for $x, y \leq -M$, $|f(x) - f(y)| \leq |f(x) - \ell| + |f(y) - \ell| < \epsilon$.

   Now we consider the closed and bounded interval $I = [-2M, 2M]$, the restriction of f on I is continuous. Therefore, by theorem 9.4, this restriction is uniformly continuous on I. So for the $\epsilon > 0$ as in above, we can find $\delta > 0$ so that $M > \delta$, and whenever $x, y \in I$ satisfy $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

   Combining the results above, for arbitrary $x, y \in \mathbb{R}$ with $|x - y| < \delta < M$. if either $x, y < -M$ or $x, y > M$, we know that $|f(x) - f(y)| < \epsilon$ is guaranteed. Otherwise, say $x \in I$, then since $M < \delta$, we can ensure that $y \in (x-\delta, x+\delta) \subset [-M-M, M+M] = I$. In this case $x, y \in I$ and so we have $|f(x) - f(y)| <$ again. This proves the uniform continuity of f on the whole $\mathbb{R}$.

2. (a) True. Suppose that we have $|f(x)-f(y)| \leq C_1|x-y|$ and $|g(x)-g(y)| \leq C_2|x-y|$ for some $C_1, C_2 \geq 0$. Note that $|(af + bg)(x) - (af + bg)(y)| \leq |af(x) - af(y)| + |bg(x) - bg(y)| \leq |a|C_1|x - y| + |b|C_2|x - y|$, so $af + bg$ is again Lipschitz.

   (b) False. A counter-example is given by $f(x) = g(x) = x$ on $\mathbb{R}$, this function is clearly Lipschitz, while $(fg)(x) = x^2$ is not Lipschitz, since $|x^2 - y^2| = |x + y| \cdot |x - y| \geq C|x - y|$ for arbitary $C > 0$ if one picks x, y sufficiently large, say $x > y > C/2$.

   (c) True. Suppose $M > 0$ is a common bound for f, g, i.e. $|f(x)| \leq M$ and $|g(x)| \leq M$. We have

   $$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - g(x)g(y)| \\ &\leq |g(x)| \cdot |f(x) - f(y)| + |f(x)| \cdot |g(x) - g(y)| \\ &\leq MC_1|x - y| + MC_2|x - y|. \end{aligned}$$

   (d) True. If $\inf_{x\in I} f > 0$, then there is some $c > 0$ so that $\inf_{x\in I} f > k$, so in particular for any $x \in I$, we have $1/f(x) < 1/k$. Then

   $$\begin{aligned} \left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| &\leq \left|\frac{f(y) - f(x)}{f(x)f(y)}\right| \\ &< \frac{1}{k^2}|f(x) - f(y)| \leq \frac{C_1}{k^2}\,|x - y|. \end{aligned}$$

(e) False. A counter-example is given by $f(x) = x^2$ on $[0, \infty)$. This is a bijective Lipschitz function on $[0, \infty)$, with inverse given by $f^{-1}(x) = \sqrt{x}$, which is non-Lipschitz. Indeed, for any $C > 0$, consider $x = \frac{1}{4C^2}$ and $y = \frac{1}{16C^2}$, then

$$|\sqrt{x} - \sqrt{y}| = \frac{1}{2C} - \frac{1}{4C} = \frac{1}{4C} > \frac{3}{16C} = \frac{3C}{16C^2} = C\left(\frac{1}{4C^2} - \frac{1}{16C^2}\right) = C|x - y|.$$

In other words, we have demonstrated for each $C > 0$, some $x, y \in [0, \infty)$ so that $|\sqrt{x} - \sqrt{y}| \leq C|x - y|$ does not hold.

3. Let f be a function of bounded variation on $\mathbb{R}$, i.e. $||f||_{BV} = \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : x_0 < x_1 < ... < x_n\}$ is finite. Assume on the contrary that $\lim_{x\to\infty} f(x)$ does not exist. We may use the Cauchy criterion for limits at infinity, i.e. $\lim_{x\to\infty} f$ exists if and only if for any $\epsilon > 0$, there exists some $M > 0$ so that for $x, y > M$, we have $|f(x) - f(y)| < \epsilon$. So if the limit does not exist, there is some $\epsilon_0 > 0$ so that for any $M > 0$ we can find $x, y > M$ so that $|f(x) - f(y)| \geq \epsilon_0$.

   Our goal is to construct an increasing sequence (x_n) so that $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$ tends to ∞ as $n \to \infty$, this will contradict to the that f has bounded variation. By the above, we may find some $x_0 < x_1$ so that $|f(x_0) - f(x_1)| \geq \epsilon_0$. Inductively, assume that we have constructed x_{2k} and x_{2k+1}, then we take $M = x_{2k+1}$ and may find $x_{2k+3} > x_{2k+2} > x_{2k+1}$ so that $|f(x_{2k+3}) - f(x_{2k+2})| \geq \epsilon_0$. This construction gives a strictly increasing sequence (x_n) so that $|f(x_{2k+1}) - f(x_{2k)}| \geq \epsilon_0$ for any $k \in \mathbb{N}$, hence

$$\begin{aligned}\sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})| &= \sum_{k=0}^{n-1} |f(x_{2k+1}) - f(x_{2k})| + \sum_{k=1}^{n} |f(x_{2k}) - f(x_{2k-1})| \\ &\geq n\epsilon_0 + \sum_{k=1}^{n} |f(x_{2k}) - f(x_{2k-1})|.\end{aligned}$$

   The latter expression approaches to infinity as $n \to \infty$. This contradicts with the assumption that $||f||_{BV}$ is finite. The case for $\lim_{x\to-\infty} f$ is identical and can be obtained by replacing $f(x)$ by $f(-x)$.

   Proposition. (Cauchy criterion at infinity) $\lim_{x\to\infty} f$ exists if and only if for any $\epsilon > 0$, there is some $M > 0$ so that for $x, y > M$, we have $|f(x) - f(y)| < \epsilon$.

   The proof is essentially the same as in proposition 7.7.

4. We take the points at which f attains its local maxima and minima. These are the points $x_k = [(k + \frac{1}{2})\pi]^{-1}$. Note that $\sin(x_k) = (-1)^k$. Consider

$$\begin{aligned}\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{n} |(-1)^k x_k - (-1)^{k-1} x_{k-1}| \\ &= \sum_{k=1}^{n} (x_k + x_{k-1}) \\ &> \sum_{k=1}^{n} \frac{1}{(k + \frac{1}{2})\pi} \\ &> \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k+1} \to \infty \text{ as } n \to \infty.\end{aligned}$$

Where in the above, we have used that the harmonic series is divergent (see example 5.4). This shows that f does not have bounded variation, as the variation can be arbitrarily large.

5. Suppose that f is differentiable with f' bounded, say $|f'(x)| < C$ for some $C > 0$. Then for $\epsilon = 1 > 0$, at each $x \in I$, we may find $\delta_x > 0$ so that if $y \in (x - \delta_x, x + \delta_x) \cap I$, we have
$$f'(x) - 1 \leq \frac{f(x) - f(y)}{x - y} \leq f'(x) + 1.$$
In particular, we have $|f(x) - f(y)| \leq (C + 1)|x - y|$ for $y \in I_x := (x - \delta_x, x + \delta_x)$. Note that however, this is not sufficient because we have to prove such inequality for all $x, y \in I$. The general case is due to compactness: if $x < y \in I$ are arbitrary, then note that $\{I_z\}_{z \in I}$ forms an open cover of the closed and bounded $[x, y]$, therefore there is a finite subcover $\{I_{z_i}\}_{i=1}^n$ of $[x, y]$. So it is possible to find $x = x_0 < x_1 < ... < x_k = y$ so that for each consecutive pair x_j, x_{j+1}, the interval (x_j, x_{j+1}) are contained in some I_{z_i} with $x_j < z_i < x_{j+1}$. Thus we have $|f(x_{j+1}) - f(x_j)| \leq |f(x_j) - f(z_i)| + |f(x_{j+1}) - f(z_i)| \leq (C + 1)|x_j - z_i| + (C + 1)|x_{j+1} - z_i| = (C + 1)|x_{j+1} - x_j|$. Putting everything together, we obtain
$$|f(x) - f(y)| \leq \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| \leq (C + 1) \sum_{j=1}^{k} |x_j - x_{j-1}| = (C + 1)|x - y|.$$

6. By assumption there are $a < b$ so that $f(a) = f(b) = 0$. Assume on the contrary that f is continuous, so by the intermediate value theorem, on each interval $(-\infty, a)$, (a, b) and (b, ∞), f must take positive or negative value. Consider the interval $[a, b]$, if f is positive on (a, b), it must attain its maximum somewhere in (a, b), and similarly for minimum if f is negative. By replacing f with $-f$ if necessary, we may assume that f is positive on (a, b). Let $z \in (a, b)$ be a point at which f attains its maximum. By intermediate value theorem, for any $f(z) > \epsilon > 0 = f(a) = f(b)$, there exists some $a < z' < z < z'' < b$ so that $f(z') = f(z'') = \epsilon$. Thus f has attained small positive values twice already. By assumption, f must be negative on $(-\infty, a)$ and (b, ∞).

   Now by assumption f must attain the maximum value twice on (a, b), say at the two points $z = z_1 < z_2$. Then for sufficiently small $\epsilon > 0$, by intermediate value theorem f must attain the value $f(z) - \epsilon$ at least three times on (a, b). More precisely, f must attain the said value at least once on each of $(a, z_1), (z_1, z_2)$ and (z_2, b). This gives a contradiction.