THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 11
1st December 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function so that $\lim_{x\to+\infty} f = L$ and $\lim_{x\to-\infty} = \ell$ both exist, show that f is uniformly continuous.

2. Let f, g be Lipschitz continuous functions on an interval I, determine whether the following statements are true. If it is true, give a proof; otherwise, provide a counterexample.
   (a) $af + bg$ is Lipschitz continuous, where $a, b \in \mathbb{R}$.
   (b) fg is Lipschitz.
   (c) If f, g are further assumed to be bounded, then fg is Lipschitz.
   (d) Suppose that $\inf_{x\in I} f > 0$, then $1/f$ is Lipschitz.
   (e) Suppose that f is injective, then f^{-1} is Lipschitz on the range of f.

3. We define a function $f : \mathbb{R} \to \mathbb{R}$ to be of bounded variation if the same condition as in definition 10.7 holds for f, without fixing the endpoints to be a, b. Show that $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist.

4. Let $f(x) = x\sin(1/x)$ for $x \in (0, \frac{2}{\pi}]$ and $f(x) = 0$ for $x = 0$. Prove that $f(x)$ is not of bounded variation.

5. Let I be an interval, a function $f : I \to \mathbb{R}$ is said to be differentiable at $c \in I$ if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, in which case the limit is denoted by $f'(c)$. It is differentiable if it is differentiable at every $c \in I$. Prove that if f is differentiable with f' bounded, then f is Lipschitz.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function which attains each value in its range exactly twice, prove that it is not continuous.