THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 10 solutions
17th November 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) It is not uniformly continuous. Let $\epsilon = 1$, then for any $\delta > 0$, consider an $x \geq 0$ to be fixed later, and $y = x + \delta/2$, so that $|x - y| < \delta$. Note that

$$|(x+\delta/2)^n - x^n| = \sum_{i=0}^{n-1} \binom{n}{i} \left(\frac{\delta}{2}\right)^{n-i} x^i \geq \frac{1}{2} n\delta x.$$

Now by Archimedean property, no matter what $\delta > 0$ is, we can find $x > 0$ large enough so that $\frac{1}{2}n\delta x \geq 1$, therefore we have shown $|f(x) - f(y)| \geq \epsilon$. f cannot be uniformly continuous.

(b) It is uniformly continuous, for $n \geq 1$. We will first prove the inequality $|x^{1/n} - y^{1/n}| \leq |x - y|^{1/n}$ for any $x, y \in [0, \infty)$. If $x = y$, then the inequality trivially holds, otherwise suppose that $x > y$, then $x = (x^{1/n} - y^{1/n} + y^{1/n})^n \geq (x^{1/n} - y^{1/n})^n + y$. Rearranging this and taking n^{th}-root (which preserves inequality as it is strictly increasing): $(x - y)^{1/n} \geq x^{1/n} - y^{1/n}$, since both sides are positive, this implies the inequality with absolute value.

Now it follows that f is uniformly continuous. Let $\epsilon > 0$, pick $\delta = \epsilon^n > 0$, then for $|x - y| < \delta$, we have

$$|x^{\frac{1}{n}} - y^{\frac{1}{n}}| \leq |x - y|^{\frac{1}{n}} < \delta^{\frac{1}{n}} = \epsilon.$$

Remark. In general, a function f satisfying $|f(x) - f(y)| \leq C|x - y|^\alpha$ for some constant $\alpha > 0$ is called a Hölder continuous function, the special case when $\alpha = 1$ is the Lipschitz condition.

2. Let $\epsilon = 1$, by uniform continuity, there is some $\delta > 0$ so that for any $x, y \in (a, b)$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < 1$. Since (a, b) is bounded, we may cover (a, b) by a finite number of open intervals (a_i, b_i) for $i = 1, ..., n$, so that $(a_i, b_i) \cap (a, b) \neq \emptyset$, $b_i = a_i + \delta$ and $a_i < a_{i+1} < b_i$ for any i. Now fix an $x_1 \in (a_1, b_1)$, for any $y \in (a, b)$, if $y \in (a_1, b_1)$, then $|x_1 - y| < b_1 - a_1 = \delta$, so $|f(y)| \leq |f(x_1) - f(y)| + |f(x_1)| < 1 + |f(x_1)|$. Otherwise $y \in (a_j, b_j)$ and one can find $x_1 < x_2 < ... < x_j = y$ so that

$x_i \in (a_i, b_i) \cap (a_{i-1}, b_{i-1})$ for $2 \le i \le j-1$. Then by construction, since both x_i and x_{i-1} lie in (a_{i-1}, b_{i-1}), they are within δ-distance from each other, hence

$$\begin{aligned}|f(y)| &\le |f(y) - f(x_1)| + |f(x_1)| \\ &\le \left|\sum_{i=2}^{j}(f(x_i) - f(x_{i-1}))\right| + |f(x_1)| \\ &\le \sum_{i=2}^{j}|f(x_i) - f(x_{i-1})| + |f(x_1)| \\ &< j - 1 + |f(x_1)| \\ &\le n - 1 + |f(x_1)| =: M.\end{aligned}$$

Since any $y \in (a, b)$ lies in some (a_j, b_j), this shows that f is bounded by the constant M.

3. Yes, first note that the composition of two uniformly continuous function is again uniformly continuous. And by Q1b, we know that $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, \infty)$. Therefore, we immediately obtain the uniform continuity of $\sqrt{f^2} = |f|$.

   To show that f itself is uniformly continuous, start with any $\epsilon > 0$, there is some $\delta > 0$ so that $|x-y| < \delta$ implies $||f(x)| - |f(y)|| < \epsilon/2$. Now for the same δ, we wish to estimate $|f(x) - f(y)|$ for $|x - y| < \delta$. If $f(x)$ or $f(y)$ is 0, or if they have the same sign, then $|f(x) - f(y)| = ||f(x)| - |f(y)|| < \epsilon/2$. Otherwise, $f(x)$ and $f(y)$ differs in signs, and so by intermediate value theorem there exists some z between x, y so that $f(z) = 0$. Now $|x - z|$ and $|y - z|$ are both smaller than δ, so we have

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f(z)| + |f(y) - f(z)| \\ &= ||f(x)| - |f(z)|| + ||f(y)| - |f(z)|| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

4. If f, g are bounded, uniformly continuous, then say $|f(x)| < M_1$ and $|g(x)| < M_2$. Then give $\epsilon > 0$, there are $\delta_1, \delta_2 > 0$ so that if $|x - y| < \delta_1$ (resp. δ_2), then $|f(x) - f(y)| < \epsilon/(2M_2)$ (resp. $g(x) - g(y)| < \epsilon/(2M_1)$. Then for $\delta = \min\{\delta_1, \delta_2\}$, for any x, y so that $|x - y| < \delta$, we have

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\ &\le |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \\ &\le |g(x)| \cdot |f(x) - f(y)| + |f(y)| \cdot |g(x) - g(y)| \\ &< M_2 \cdot \frac{\epsilon}{2M_2} + M_1 \cdot \frac{\epsilon}{2M_1} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

   Boundedness cannot be dropped, e.g. as we have seen x^2 is not uniformly continuous on $\mathbb{R}$ despite x is.

5. Let f be a continuous periodic function with period $p > 0$, then the restriction of f onto $[0, 2p]$ is uniformly continuous because $[0, 2p]$ is compact. So for arbitrary $\epsilon > 0$, there is a $\delta > 0$ so that whenever $x, y \in [0, 2p]$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Now we take $\delta' = \min\{p, \delta\}$, we consider now arbitrary $x \le y \in \mathbb{R}$ with $|x - y| < \delta$. Since $\bigcup_{n\in\mathbb{Z}}[n, (n+1)p] = \mathbb{R}$, there exists some $k \in \mathbb{Z}$ so that $x \in [np, (n+1)p]$, in order words $x - np \in [0, p]$. Then it follows that $y - np \in [x, x+\delta) \subset [0, 2p]$. Now $|y - np - (x - np)| = |x - y| < \delta$, so by uniform continuity of f on $[0, 2p]$, we have $|f(x - np) - f(y - np)| < \epsilon$, by periodicity this implies $|f(x) - f(y)| < \epsilon$.

6. (a) f is uniformly continuous if and only if for all $\epsilon > 0$, there exists δ such that for all $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

   This is equivalent to: for all $\epsilon > 0$, there exists $\delta > 0$ such that $\omega_f(\delta) = \sup\{|f(x) - f(y)| : x, y \in A, |x - y| < \delta\} < \epsilon$.

   Notice that $\omega_f(\delta)$ is monotone increasing, as the subset we are taking supremum over is monotone increasing as δ increases. Therefore, the latter is equivalent to requiring $\omega_f(\delta') < \epsilon$ for all $0 < \delta' < \delta$, by suitably replacing ϵ if necessary. This condition is equivalent to $\lim_{\delta\to 0}\omega_f(\delta) = 0$.

   (b) $\omega_f(0) = 0$ always holds for all any function f, as we are taking supremum over a singleton $\{0\}$. For any $\delta > 0$, we will prove that $\omega_f(\delta) = 2$. This is due to Archimedean property, there always exists $N \in \mathbb{N}$ large enough so that $\frac{1}{(2N+1/2)\pi} - \frac{1}{(2N+1+1/2)\pi} < \delta$. But then

$$\sin\left(\frac{1}{(2N+1/2)\pi}\right) - \sin\left(\frac{1}{(2N+1+1/2)\pi}\right) = 1 - (-1) = 2.$$

   By triangle inequality, $\sup\{|f(x) - f(y)| : |x - y| < \delta\} \le 2\sup\{|f(x)| : x \in (0, 1)\} = 2$, we see that the supremum is achieved. So $\omega_f(\delta) = 2$.

   Now $\lim_{\delta\to 0}\omega_f(\delta) \neq 0$, we conclude that $f = \sin(1/x)$ is not uniformly continuous.

   (c) Let f be a uniformly continuous function on $\mathbb{R}$, we will prove that ω_f is subadditive, i.e. $\omega_f(s+t) \le \omega_f(s) + \omega_f(t)$. Consider the following sets

$$A = \{|f(x) - f(y)| : |x - y| < s + t\}.$$
$$B = \{|f(x) - f(u)| + |f(u) - f(y)| : |x - u| < s, |y - u| < t\}.$$
$$C = \{|f(x) - f(u)| + |f(v) - f(y)| : |x - u| < s, |y - v| < t\}.$$

   Then we have $A \subset B \subset C$, where the first inclusion is due to triangle inequality, and the second inclusion is seen by specializing $v = u$. Therefore $\sup A \le \sup B \le \sup C$, but then $\sup A = \omega_f(s+t)$ and $\sup C = \omega_f(s) + \omega_f(t)$ (recall that supremum of sum of two sets is the sum of their supremums). Therefore we have proven the subadditivity of ω_f.

   Now take $s = a - b$ and $t = b$ for some $a \ge b$, then $\omega_f(s+t) = \omega_f(a) \le \omega_f(a-b) + \omega_f(b)$, and so $|\omega_f(a) - \omega_f(b)| < \omega(|a - b|)$. This implies that ω_f is uniformly continuous, since given any $\epsilon > 0$, continuity at 0 implies that for any $\epsilon > 0$, there exists $\delta > 0$ so that $0 < \delta' < \delta$ implies $\omega_f(\delta') < \epsilon$. Now take $\delta/2 > 0$, if $|a - b| < \delta/2$, we have $|\omega_f(a) - \omega_f(b)| < \omega_f(|a - b|) < \omega_f(\delta/2) < \epsilon$.

Remark. Note that in the above, we only used (i) ω_f continuous at 0, (ii) ω_f is subadditive and (iii) ω_f is strictly increasing. By the above proof, any function satisfying these three properties would be uniformly continuous.

7. ($\Rightarrow$) Suppose that f is uniformly continuous, then f restricted to each of $[-1,0)$ and $(0,1]$ is uniformly continuous as well. By continuous extension theorem, this holds if and only if $f|_{[0,1)}$ (resp. $f|_{(-1,0]}$) can be extended to the endpoints so that $f_1 = f|_{[0,1]}$ (resp. $f_2 = f|_{[-1,0]}$) is continuous. Recall that such extensions must satisfies $f_1(0) = \lim_{x\to 0^+} f(x)$ and $f_2(0) = \lim_{x\to 0^-} f(x)$. So the one-sided limits exist. Suppose that they differ, then we can find a sequence (x_n) in $(0,1]$ so that $\lim x_n = 0$; and a sequence (y_n) in $[-1,0)$ so that $\lim y_n = 0$. Then we know $\lim f(x_n) \neq \lim f(y_n)$, it follows that for ϵ chosen to be say $|f_1(0) - f_2(0)|/2$, for any $\delta > 0$, then there exists some $N \in \mathbb{N}$ so that for N large enough, $|x_n - y_n| < \delta$ and $|f(x_n) - f(y_n)| \geq \epsilon$ for $n \geq N$. This contradicts with uniform continuity of f. So $f_1(0) = f_2(0)$ must hold.

   ($\Leftarrow$) Suppose that f satisfies $\lim_{x\to 0^+} f = \lim_{x\to 0^-} f$, then f can be extended to a continuous function on $[-1,1]$. Since $[-1,1]$ is compact, it follows that the extension of f to $[-1,1]$ is uniformly continuous, whose restriction onto $[-1,0)\cup(0,1]$, i.e. f itself, is also clearly uniformly continuous.

8. Take $f(x) = \sin(\pi x)$, then $f(n) = 0$ for any $\in \mathbb{Z}$, so $\lim f(n) = 0$. But $\lim_{x\to\infty} f(x) \neq 0$ since it is periodic.

   Now suppose further that $g(x) = f(x^2)$ is uniformly continuous. For any $\epsilon > 0$, by assumption there is some $N_1 \in \mathbb{N}$ so that $|f(n)| = |g(\sqrt{n})| < \epsilon/2$ for any $n \geq N_1$. Now by uniform continuity, there is also some $\delta > 0$ so that whenever $|x - y| < \delta$, we have $|g(x) - g(y)| < \epsilon/2$. Since $\lim(\sqrt{n+1} - \sqrt{n}) = 0$, there exists N_2 so that for $n \geq N_2$, we can guarantee $\sqrt{n+1} - \sqrt{n} < \delta$. Now we take $N = \max\{N_1, N_2\}$, and any $x \geq N > 0$, we have $\sqrt{x} \in [\sqrt{n}, \sqrt{n+1})$ for some $n \geq N$. Then by construction, $|\sqrt{x} - \sqrt{n}| < \delta$, we conclude

$$\begin{aligned} |f(x)| = |g(\sqrt{x})| &\leq |g(\sqrt{x}) - g(\sqrt{n})| + |g(\sqrt{n})| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$