THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 8 solutions
16th November 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. Let $p(x) = a_{2k+1}x^{2k+1} + ... + a_1x + a_0$ be a polynomial of odd degree, without loss of generality, we may assume $a_{2k+1} = 1$. This is because $p(x)$ has α as a root if and only if $q(x) = p(x)/a_{2k+1}$ has α as a root, the latter has leading coefficient 1.

   Now we know that

$$\lim_{x\to\infty} \left| \frac{a_{2k}}{x} + \frac{a_{2k-1}}{x^2} + \cdots + \frac{a_1}{x^{2k}} + \frac{a_0}{x^{2k+1}} \right| = 0.$$

   Therefore, picking $\epsilon = 1/2$, we can find $M > 0$ so that for $x \geq M$, the expression above is bounded above by 1. In particular, after rearranging, M satisfies the inequality

$$\frac{1}{2}M^{2k+1} \geq a_{2k}M^{2k} + ... + a_1M + a_0 \geq -\frac{1}{2}M^{2k+1}. \tag{1}$$

   So we have $p(M) \geq M^{2k+1} - \frac{1}{2}M^{2k+1} = \frac{1}{2}M^{2k+1} > 0$.

   Similarly, the limit is also 0 as $x \to -\infty$, so for $\epsilon = 1/2$, we have some $m < 0$ so that (1) holds for m in place of M. So again $p(m) \leq m^{2k+1} + \frac{1}{2}m^{2k+1} = \frac{3}{2}m^{2k+1} < 0$ since m is negative.

   By intermediate value theorem, there exists some $c \in (m, M)$ so that $p(c) = 0$.

2. By assumption on the limits, there exists $M > 0$ so that for $x \geq M$ or $x \leq -M$, we have $|f(x)| < 1$. And over $[-M, M]$, the restriction of f is a continuous function over a compact subset, hence by theorem 8.10 its image is also compact, which implies that it is bounded. So the original function f is globally bounded over $\mathbb{R}$.

   Since f is bounded, both its supremum and infimum exist. If $\sup f = \inf f = 0$, then f is the constant 0 function, in this case f does attain its maximum or minimum. Now assume otherwise that $\sup f \neq \inf f$. At least one of $\sup f$ or $\inf f$ is non-zero. By replacing f with $-f$ if necessary, we may assume that $S = \sup_{\mathbb{R}} f > 0$. By assumptions on the limits, there exists $M > 0$ large enough so that for $x \geq M$ or $x \leq -M$, we have $f(x) \leq |f(x)| < S/2$. Now consider some $0 < \epsilon < S/2$, then property of supremum, there is some $x \in \mathbb{R}$ so that $f(x) > S - \epsilon \geq S - S/2 = S/2$. By the above, such x cannot be in the range $(-\infty, -M] \cup [M, \infty)$, i.e. $x \in [-M, M]$. In particular, this implies that $S = \sup_{[-M,M]} f$. Since the supremum must be attained in a compact domain, there is some $x_0 \in [-M, M]$ with $S = f(x_0)$ as desired.

3. On $[1, \infty)$, note that $1/x \leq 1$. Therefore given $\epsilon > 0$, we can pick $\delta = \epsilon$, then if $|x - y| < \delta$, we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x - y}{xy}\right| \leq |x - y| < \delta = \epsilon.$$

On $(0, \infty)$, we shall prove that $1/x$ is not uniformly continuous. Take $\epsilon = 1$, then for any $\delta > 0$, there exists some $N \in \mathbb{N}$ so that $\delta > \frac{1}{N}$, we may consider $x = \frac{1}{N}, y = \frac{1}{2N}$, note that $|x - y| = \frac{1}{2N} < \delta$, but

$$\left|\frac{1}{1/N} - \frac{1}{1/(2N)}\right| = N \geq 1.$$