THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 6 solutions
10th November 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. Let $f : A \to \mathbb{R}$ be a non-negative function with $\lim_{x\to c} f = L$, then we claim that $\lim_{x\to c} \sqrt{f} = \sqrt{L}$. We will deal with the cases when $L = 0$ and $L \neq 0$ separately. Denote $A_{\delta,c} := (c-\delta, c+\delta) \cap A \setminus \{c\}$.

   First suppose that $L = 0$, then for any $\epsilon > 0$, there exists some $\delta > 0$ so that in the range of $A_{\delta,c}$, we have $|f(x)| < \epsilon^2$. Therefore, for all $x \in A_{\delta,c}$, $|\sqrt{f(x)}| < \sqrt{\epsilon^2} = \epsilon$.

   Now suppose $L \neq 0$, since $f(x) \geq 0$, we have $\lim_{x\to c} f(x) = L \geq 0$, this implies $L > 0$. Now by the limit, we know there is some $\delta_1 > 0$ so that on $A_{\delta_1,c}$, we have $|f(x) - L| < 3L/4$, in particular $f(x) > L/4$. Now choose any $\epsilon > 0$, by convergence of f there exists some $\delta_2 > 0$ so that on $A_{\delta_2,c}$, we have $|f(x) - L| < \frac{3}{2}\sqrt{L}\epsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$, then for any $x \in A_{\delta,c}$,

$$|\sqrt{f(x)} - \sqrt{L}| = \frac{|f(x) - L|}{\sqrt{f(x)} + \sqrt{L}} \leq \frac{|f(x) - L|}{\frac{3}{2}\sqrt{L}} < \epsilon.$$

2. Let $f(x) = |x|^{-\frac{1}{2}}$, for any $M > 0$, we may simply pick $\delta = 1/M^2 > 0$, then for any x in the range of $0 < |x| < \delta$, we have

$$\frac{1}{\sqrt{|x|}} > \frac{1}{\sqrt{\delta}} = M.$$

   In particular, this holds for x either in $(0, \delta)$ or $(-\delta, 0)$, hence the two-sided limits are both $+\infty$.

3. Suppose that $\lim_{x\to c} f(x) = L > 0$ and $\lim_x g(x) = \infty$. Then for $\epsilon = L/2 > 0$, we can find $\delta_1 > 0$ so that in the range of $0 < |x - c| < \delta_1$, we have $|f(x) - L| < L/2$, in particular, this implies $f(x) > L/2$. Now given arbitrary $M > 0$, since the limit of $g(x)$ is ∞, we can find $\delta_2 > 0$ so that for x in the range of $0 < |x - c| < \delta_2$, we have $g(x) > 2M/L$.

   Now we set $\delta = \min\{\delta_1, \delta_2\}$, then for x in the range of $0 < |x - c| < \delta$, both $f(x) > L/2$ and $g(x) > 2M/L$ are satisfied. Hence $f(x)g(x) > M$, this concludes the proof of $\lim_{x\to c} f(x)g(x) = \infty$.

   For a counter-example to the second claim, simply take $f(x) = x^2$ and $g(x) = 1/x^2$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$, while $f(x)g(x) = 1$ for $x \neq 0$, and so $\lim_{x\to 0} f(x)g(x) = 1 \neq \infty$.