THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 5 solutions
1st November 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. Let $x_n = \sqrt{n}$, first we will show that $\lim |\sqrt{n+1} - \sqrt{n}| = 0$. Given $\epsilon > 0$, by Archimedean property we can find $N \in \mathbb{N}$ so that $N > \frac{1}{4\epsilon^2} - 1$, then for $n \geq N$,

$$|\sqrt{n+1} - \sqrt{n}| = \frac{n+1-1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n+1}} \leq \frac{1}{2\sqrt{N+1}} < \epsilon.$$

Next, to see that (x_n) is not Cauchy, it suffices to show that it is unbounded. This is clear because for $n \geq M^2$, we have $\sqrt{n} \geq \sqrt{M^2} = M$ for arbitrary $M > 0$.

2. For arbitrary $m, n \in \mathbb{N}$ with $m > n$, we have

$$|x_m - x_n| = \left|\sum_{j=n}^{m-1}(x_{j+1} - x_j)\right| \leq \sum_{j=n}^{m-1}|x_{j+1} - x_j| < \sum_{j=n}^{m-1} r^j = \frac{r^n - r^m}{1-r} < \frac{r^n}{1-r}.$$

Recall that since $0 < r < 1$, we know that $\lim \frac{r^n}{1-r} = 0$ (see for example, tutorial 2 Q6; one can prove this by using Bernoulli's inequality or just monotone convergence theorem). Therefore, given $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that for $n \geq N$, we have $0 < \frac{r^n}{1-r} < \epsilon$. Then for $m > n \geq N$, from the calculation above,

$$|x_m - x_n| < \frac{r^n}{1-r} < \epsilon.$$

3. (a) Consider

$$\left|\frac{2x+3}{4x-9} - 3\right| = \left|\frac{2x+3-12x+27}{4x-9}\right| = \frac{10|x-3|}{|4x-9|}.$$

For any $\epsilon > 0$, we pick $\delta = \min\{\epsilon/10, 1/2\}$. Then for x in the range of $0 < |x-3| < \delta$, in particular, we have $5/2 < x < 7/2$. And so $1 < |4x-9|$.

$$\left|\frac{2x+3}{4x-9} - 3\right| = \frac{10}{|4x-9|} \cdot |x-3| < 10\delta \leq \epsilon.$$

(b) Consider

$$\left|\frac{x^2-3x}{x+3} - 2\right| = \left|\frac{x^2-3x-2x-6}{x+3}\right| = \frac{|x+1|}{|x+3|} \cdot |x-6|.$$

For any $\epsilon > 0$, we pick $\delta = \min\{\epsilon, 1\}$, then for x in the range of $0 < |x-6| < \delta$, in particular we have $5 < x < 7$, so that $\frac{|x+1|}{|x+3|} < 1$ is always satisfied. Now,

$$\left|\frac{x^2-3x}{x+3} - 2\right| = \frac{|x+1|}{|x+3|} \cdot |x-6| < \delta \leq \epsilon.$$