THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 4 solutions
11th October 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. ($\Longrightarrow$) If $\lim z_n = L$ exists, then the subsequences $z_{2n-1} = x_n$ and $z_{2n} = y_n$ are also convergent, with $\lim z_{2n-1} = \lim z_{2n} = L$. So they have the same limit.

   ($\Longleftarrow$) Suppose that $\lim x_n = \lim y_n = L$, we will show that $\lim z_n = L$ from definition. Given $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ so that for $n \geq N_1$ (resp. $n \geq N_2$), we have $|x_n - L| < \epsilon$ (resp. $|y_n - L| < \epsilon$).

   If we take $N = \max\{2N_1, 2N_2\}$, for any $m \geq N$, if $m = 2n-1$ is odd, then $2n-1 \geq 2N_1$ implies that $n \geq N_1$, and hence $|z_{2n-1} - L| = |x_n - L| < \epsilon$. And if $m = 2n$ is even, we have $2n \geq 2N_2$, so that $n \geq N_2$, and therefore $|z_{2n} - L| = |y_n - L| < \epsilon$. This proves that $\lim z_n = L$.

2. Suppose that (x_n) is unbounded, then for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ so that $n_k > n_{k-1}$ and $x_{n_k} > k$. Then for this subsequence, first of all, every term is non-zero, so $1/x_{n_k}$ is well-defined, and by construction $0 < 1/x_{n_k} < 1/k$. Therefore, by squeeze theorem $\lim 1/x_{n_k} = 0$.

3. Since $s := \sup\{x_n : n \in \mathbb{N}\}$ does not belong to the sequence itself, given any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ so that $n_k > n_{k-1}$ and $x_{n_k} > s - \frac{1}{k}$. Then by construction $\frac{1}{k} > |s - x_{n_k}| > 0$, so by squeeze theorem x_{n_k} is convergent with limit equals to s.