THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 1 solutions
13th September 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. We claim that $\sup S = 1$, first note that 1 is an upper bound of S: for any $\frac{1}{n} - \frac{1}{m} \in S$, $1 \geq \frac{1}{n} \geq \frac{1}{n} - \frac{1}{m}$. Now by ϵ-characterization, we pick any $\epsilon > 0$, then Archimedean property asserts that there is some $m \in \mathbb{N}$ so that $\epsilon > \frac{1}{m}$, then $1 - \epsilon < 1 - \frac{1}{m} \in S$. Hence 1 is indeed the least upper bound, i.e. supremum.

   As for infimum, note that for any $\frac{1}{n} - \frac{1}{m} \in S$, its negative $-\frac{1}{n} + \frac{1}{m}$ is also an element of S. In other words, we have $S = -S$, so $\inf(S) = \inf(-S) = -\sup(S) = -1$.

2. Recall that $\sup_{x\in X} f(x)$ is defined to be $\sup f(X)$ where $f(X)$ denotes the set of images of f. To prove the inequality, we will prove the strict inequality version instead, i.e. we will show $\sup_{x\in X}(f+g) - \epsilon < \sup_{x\in X} f + \sup_{x\in X} g$. Consider the LHS of the inequality, by ϵ-characterization of supremum, we can find some $x_0 \in X$ so that $\sup_{x\in X}(f+g) - \epsilon < f(x_0) + g(x_0) \leq \sup_{x\in X} f + \sup_{x\in X} g$. Where the second inequality follows from that $\sup$ is an upper bound, so will bound any particular value above.

   The statement of infimum can be obtained by taking functions $\tilde{f} = -f$ and $\tilde{g} = -g$. Then $\inf_{x\in X} \tilde{f} = \inf_{x\in X}(-f) = \inf(-f(X)) = -\sup f(X) = -\sup_{x\in X} f$. So the inequality for supremum implies that

$$\inf_{x\in X}(\tilde{f} + \tilde{g}) = -\sup_{x\in X}(f+g) \geq -\sup_{x\in X} f - \sup_{x\in X} g = \inf_{x\in X} \tilde{f} + \inf_{x\in X} \tilde{g}.$$