$A \in M_{n\times n}(\mathbb{C})$. A is normal $\Leftrightarrow \exists P \in U(n)$ s.t $P^*AP = D$

$A \in M_{n\times n}(\mathbb{R})$ A is sym $\Leftrightarrow \exists P \in O(n)$ s.t $P^TAP = D$

Given $A \in M_{n\times n}(\mathbb{C})$, Find $P \in U(n)$ s.t $P^*AP = D$?

① $f_A(t) = \det(A - t\cdot I_n)$ $\lambda_1 \cdots \lambda_k$

② for each λ_j . $E_{\lambda_j} = N(A - \lambda_j \cdot I_n)$

find basis β_j for E_{λ_j} $j = 1 \cdots k$

③ Apply G.S process on β_j to obtain β_j' (Orthogonal)

④ normalize β_j' to β_j'' (Orthonormal)

⑤ $\beta = \beta_1'' \cup \cdots \cup \beta_k''$ is o.n. basis for V consisting of eig vec. of A.

$\beta = \{v_1 \cdots v_n\}$

$$A \underbrace{[v_1 \cdots v_n]}_{P} = [v_1 \cdots v_n] \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}}_{D}$$

$$AP = P \cdot D \overset{P^*P = I}{\Leftrightarrow} P^*AP = D$$

$$A = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{find } P \in U(n) \quad P^* A P = D.$$

$$f_A(t) = \det \begin{pmatrix} i-t & & \\ & -t & -1 \\ & 1 & -t \end{pmatrix} = (i-t)\cdot\left((-t)^2+1\right)$$

$$= -(t-i)(t-i)(t+i) = -(t-i)^2(t+i)$$

- for $\lambda_1 = i$ $\qquad E_{\lambda_1} = N(A - \lambda_1 I_3)$

$$A - \lambda_1 I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$

GS & normalization $\quad u_1 = v_1 \qquad u_2 = v_1 / \|v_1\| = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} / \sqrt{2}$

- for $\lambda_2 = -i$

$$A - \lambda_2 I_3 = \begin{pmatrix} 2i & 0 & 0 \\ 0 & i & -1 \\ 0 & 1 & i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \qquad u_3 = v_3 / \|v_3\| = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} / \sqrt{2}$$

$$\beta = \{u_1 \ u_2 \ u_3\} \qquad A \underbrace{[u_1 \ u_2 \ u_3]}_{P} = [u_1 \ u_2 \ u_3] \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix}}_{D}$$

$$P^* A P = \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix}$$

Cor.

If $F = \mathbb{C}$. then T is normal iff $T^* = g(T)$ for some poly g.

Pf. ($\Leftarrow$) $T^* = g(T)$ $\quad T^*T = g(T)\circ T = T \circ g(T) = TT^*$

($\Rightarrow$) $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$

$T^* = (\lambda_1 T_1 + \cdots + \lambda_k T_k)^* = \overline{\lambda_1} T_1^* + \cdots + \overline{\lambda_k} T_k^*$

$= \overline{\lambda_1} T_1 + \cdots + \overline{\lambda_k} \cdot T_k$

Lagrange Interpolation $\exists$ poly g st $g(\lambda_j) = \overline{\lambda_j}$ $\forall j$

$g(T) = g(\lambda_1 T_1 + \cdots + \lambda_k T_k)$

$= g(\lambda_1)\cdot T_1 + \cdots + g(\lambda_k)\cdot T_k$

$= \overline{\lambda_1} T_1 + \cdots + \overline{\lambda_k} T_k$

$= T^*$

$$A = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad T = L_A : \mathbb{C}^3 \to \mathbb{C}^3 \quad \text{normal.}$$

$$x \mapsto Ax$$

find poly g. s.t $T^* = g(T)$?

- $f_T(t) = f_A(t) = -(t-i)^2(t+i)$ $\quad \begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$

- $$\begin{cases} f_1(x) = \dfrac{x-\lambda_2}{\lambda_1-\lambda_2} \\ f_2(x) = \dfrac{x-\lambda_1}{\lambda_2-\lambda_1} \end{cases} \qquad f_i(\lambda_j) = \delta_{ij}$$

Let
$$\begin{aligned} g(x) &= \overline{\lambda_1} \cdot f_1(x) + \overline{\lambda_2} f_2(x) \\ &= \overline{\lambda_1} \cdot \frac{x-\lambda_2}{\lambda_1-\lambda_2} + \overline{\lambda_2} \cdot \frac{x-\lambda_1}{\lambda_2-\lambda_1} \\ &= -i \cdot \frac{x+i}{2i} + i \cdot \frac{x-i}{-2i} \\ &= -x \end{aligned}$$

$g(\lambda_1) = \overline{\lambda_1} \qquad g(\lambda_2) = \overline{\lambda_2}$

- $T = L_A \qquad g(T) = T^*$?

$$g(A) = A^* \,!$$

Sec 6.4 Q16

16. Prove the *Cayley–Hamilton theorem* for a complex $n \times n$ matrix A. That is, if $f(t)$ is the characteristic polynomial of A, prove that $f(A) = O$. *Hint:* Use Schur's theorem to show that A may be assumed to be upper triangular, in which case

$$f(t) = \prod_{i=1}^{n} (A_{ii} - t).$$

Now if $\mathsf{T} = \mathsf{L}_A$, we have $(A_{jj}\mathsf{I} - \mathsf{T})(e_j) \in \text{span}(\{e_1, e_2, \ldots, e_{j-1}\})$ for $j \geq 2$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard ordered basis for C^n. (The general case is proved in Section 5.4.)

let $T = L_A$. By Schur's thm, $\exists$ orthonormal basis γ of $\mathbb{C}^n$ st $[T]_\gamma$ is upper triangular.

let $\beta = \{e_1, \cdots, e_n\}$ be the standard basis for $\mathbb{C}^n$

then $[T]_\gamma = [I]_\beta^\gamma \, A \, [I]_\gamma^\beta = Q^{-1} A Q$ for some invertible Q.

Obviously $f(Q^{-1}AQ) = 0 \iff f(A) = 0$

Thus it suffice to prove the result for all upper-triangular A.

Suppose A is upper-triangular. let $T = L_A$

The char poly of A is $f(t) = \det(A - tI_n) = \prod_{i=1}^{n} (A_{ii} - t)$

For any $j \geq 2$ $\quad (A_{jj} I - T) e_j = A_{jj} e_j - (A_{1j} e_1 + \cdots + A_{jj} e_j) \in \text{span}(\{e_1, \cdots, e_{j-1}\})$

Since A is upper triangular. for any $i < j$

$(A_{jj} I - T) e_i = A_{jj} e_i - T e_i \in \text{span}(\{e_1, \cdots, e_i\}) \subset \text{span}(\{e_1, \cdots, e_{j-1}\})$

Thus for any $x \in \text{span}(\{e_1, \cdots, e_n\}) = \mathbb{C}^n$,

$$(A_{nn} I - T) x \in \text{span}(\{e_1, \cdots, e_{n-1}\})$$

$$(A_{n-1\,n-1} I - T)(A_{nn} I - T) x \in \text{span}(\{e_1, \cdots, e_{n-2}\})$$

$$\vdots$$

$$(A_{22} I - T) \cdots (A_{nn} I - T) x \in \text{span}(\{e_1\})$$

and $(A_{11} I - T) \cdots (A_{nn} I - T) x = 0$ i.e. $f(T)(x) = 0$

$\therefore f(A) = [f(T)]_\beta = [T_0]_\beta = O_{n \times n}$

Sec 6-5 Q15

15. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that

(a) $\mathsf{U}(\mathsf{W}) = \mathsf{W}$;
(b) $\mathsf{W}^{\perp}$ is U-invariant.

(a) W is U-inv. $U(W) \subset W$ $\qquad U^* U = I$

let $\{w_1, \dots w_k\}$ be basis for W

$a_1 U(w_1) + \dots + a_k \cdot U(w_k) = 0 \qquad a_j \in F$

$U(a_1 w_1 + \dots + a_k w_k) = 0$

$U^* U(a_1 w_1 + \dots + a_k w_k) = U^*(0) = 0$

$a_1 w_1 + \dots + a_k w_k = 0 \Rightarrow a_j = 0 \ \forall j$

$\dim(W) = \dim(U(W))$

$W = U(W)$.

(b) $\forall x \in W^\perp \qquad \forall y \in W. \ \exists z \in W. \text{ st } y = U(z)$

$\langle U(x), y \rangle = \langle x, U^*(y) \rangle = \langle x, U^*(U(z)) \rangle = \underset{\substack{\shortparallel \\ 0}}{\langle x, z \rangle} \qquad \forall y \in W$

$U(x) \in W^\perp$

See 6.6 Q6

6. Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

let T be a projection of a fin-dim inner product space. V

we need to show $R(T)^{\perp} = N(T)$ $\quad N(T)^{\perp} = R(T)$

$\forall x \in R(T)^{\perp}$, $\langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle = \langle x, \underset{R(T)}{\underset{\uparrow}{TT^*(x)}} \rangle = 0$

$\therefore x \in N(T)$ i.e. $R(T)^{\perp} \subset N(T)$

$\forall x \in N(T)$, $T(x) = 0 = 0x$ so $T^*(x) = \bar{0}x = 0$ (since T is normal)

$\langle x, T(y) \rangle = \langle T^*(x), y \rangle = \langle 0x, y \rangle = 0$ $\quad \forall y \in V$

$\therefore x \in R(T)^{\perp}$ i.e. $N(T) \subset R(T)^{\perp}$

Hence $R(T)^{\perp} = N(T)$

Since the space is fin-dim (by Sec 6.2 Q13(c))

$N(T)^{\perp} = (R(T)^{\perp})^{\perp} = R(T)$

So T is an orthogonal projection.