Midterm 1 Q2

2. Let $p_0(x) = x + 1$. Consider the following mapping

$$T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$
$$p(x) \mapsto \begin{pmatrix} p(0) & p'(1) \\ (p_0 \cdot p)'(0) & \int_0^1 p(t)dt \end{pmatrix}$$

Let $\beta = \{1, x, x^2\}$ and $\gamma = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ be bases for $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ respectively.

(a) Show that T is a linear transformation.

(b) Compute $[T]_\beta^\gamma$. Please show your steps.

(c) Use the rank–nullity theorem to determine whether T is one-to-one. Please explain your answer with details.

(b)

$$T(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 0 & 1 \\ 1 & 1/2 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & \frac{1}{3} \end{pmatrix} = \frac{7}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

OR Let $\gamma' = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ be standard ordered basis for $M_{2\times 2}(\mathbb{R})$

$$[T]_\beta^\gamma = [I_{M_{2\times2}(\mathbb{R})} \circ T]_\beta^\gamma = [I_{M_{2\times2}(\mathbb{R})}]_{\gamma'}^\gamma \cdot [T]_\beta^{\gamma'}$$

$$= \left([I_{M_{2\times2}(\mathbb{R})}]_\gamma^{\gamma'}\right)^{-1} \cdot [T]_\beta^{\gamma'}$$

$$\because I_n = [I_{M_{2\times2}(\mathbb{R})}]_\gamma^\gamma = [I_{M_{2\times2}(\mathbb{R})}]_{\gamma'}^\gamma \cdot [I_{M_{2\times2}(\mathbb{R})}]_\gamma^{\gamma'}$$

$$\therefore \left([I_{M_{2\times2}(\mathbb{R})}]_\gamma^{\gamma'}\right)^{-1} = [I_{M_{2\times2}(\mathbb{R})}]_{\gamma'}^\gamma$$

Sec 2.5 Q13

13.[†] Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \le j \le n,$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

Given basis β

- For any basis β', $[I_V]_{\beta'}^{\beta}$ change of coord matrix is invertible
- For any invertible Q, $\exists \beta'$ basis for V. s.t. $[I]_{\beta'}^{\beta} = Q$?

1-1 correspondence between basis β' and invertible Q.

- β' is L.I.

Consider $a_1 \cdot x'_1 + \cdots + a_n \cdot x'_n = 0$

$$0 = \sum_{j=1}^{n} a_j \cdot x'_j = \sum_j a_j \cdot \sum_i Q_{ij} x_i = \sum_i \left(\sum_j Q_{ij} a_j \right) x_i$$

Since β is basis . L.I, $\sum_j Q_{ij} a_j = 0 \quad i = 1, \ldots, n$

i.e $Q \cdot \vec{a} = \vec{0} \qquad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Since Q invertible, $\vec{a} = \vec{0}$ $\therefore a_1 = \cdots = a_n = 0$. β' L.I

- $\beta' \subset V$. $\dim(V) = n = |\beta'|$ Thus β' is basis

- $$\left([I_V]_{\beta'}^{\beta} \right)_{ij} = \left([I_V(x'_j)]_{\beta} \right)_i = \begin{pmatrix} Q_{1j} \\ \vdots \\ Q_{nj} \end{pmatrix}_i = Q_{ij}$$

11. Let V and W be finite-dimensional vector spaces over F, and let ψ_1 and ψ_2 be the isomorphisms between V and V^{**} and W and W^{**}, respectively, as defined in Theorem 2.26. Let $T\colon V \to W$ be linear, and define $T^{tt} = (T^t)^t$. Prove that the diagram depicted in Figure 2.6 commutes (i.e., prove that $\psi_2 T = T^{tt}\psi_1$).

dual space

$$\begin{array}{ccc}
v \in V & \xrightarrow{\ T\ } & W \ni w \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
\psi_1(v) = \hat{v} \in V^{**} & \xrightarrow{\ T^{tt}\ } & W^{**} \ni \psi_2(w) = \hat{w}
\end{array}$$

T^t is T^*

different notation

dual map / transpose

Figure 2.6

- Recall that V is isomorphic to V^{**}

$$\psi: V \to V^{**} \quad \text{where} \quad \varphi(v): V^* \to F \qquad \psi \text{ is isomorphism}$$
$$v \mapsto \varphi(v) \qquad\qquad f \mapsto \varphi(v)(f) := f(v)$$

So $\begin{cases} \text{Let } \hat{v} \text{ denotes } \psi_1(v) \text{ for any } v \in V \\ \text{Let } \hat{w} \text{ denotes } \psi_2(w) \text{ for any } w \in W \end{cases}$

- Let $\begin{cases} \beta = \{v_1 \cdots v_n\} \text{ be basis for } V \\ \gamma = \{w_1 \cdots w_m\} \text{ be basis for } W, \end{cases}$ and $\begin{cases} \beta^* = \{f_1 \cdots f_n\} \text{ dual basis of } \beta \\ \gamma^* = \{g_1 \cdots g_m\} \text{ dual basis of } \gamma \end{cases}$

Consider $\begin{cases} \beta^{**} = \{\hat{v}_1 \ \cdots \ \hat{v}_n\} \\ \gamma^{**} = \{\hat{w}_1 \ \cdots \ \hat{w}_m\} \end{cases}$ $\quad \begin{aligned} \hat{v}_i &= \varphi_1(v_i) \quad i = 1 \cdots n \\ \hat{w}_j &= \varphi_2(w_j) \quad j = 1 \cdots m \end{aligned}$

Then $\begin{cases} \hat{v}_i(f_j) = f_j(v_i) = \delta_{ij} \Rightarrow \beta^{**} \text{ is dual basis of } \beta^* \\ \hat{w}_i(g_j) = g_j(w_i) = \delta_{ij} \Rightarrow \gamma^{**} \text{ is dual basis of } \gamma^* \end{cases}$

- $T: V \to W \qquad T^*: W^* \to V^* \qquad T^{**}: V^{**} \to W^{**}$
$\quad\ v \mapsto w \qquad\qquad g \mapsto g \circ T \qquad\qquad \hat{v} \mapsto \hat{v} \circ T^*$

$$W^* \xrightarrow{T^*} V^* \xrightarrow{\hat{v}} F$$

- Let $A = [T]_\beta^\gamma$

- For $j = 1 \cdots n$

$$\begin{cases} \psi_2 \circ T(v_j) = \psi_2\left(\sum_{i=1}^m A_{ij} \cdot w_i\right) = \sum_{i=1}^m A_{ij}\, \psi_2(w_i) = \sum_{i=1}^m A_{ij}\, \hat{w}_i \\ T^{**} \circ \psi_1(v_j) = T^{**}(\hat{v}_j) = \hat{v}_j \circ T^* \end{cases}$$

Claim: $\sum_{i=1}^{m} A_{ij} \cdot \hat{w}_i = \hat{v}_j \circ T^* \in W^{**}$ $\quad W^* \to F$

To prove that they agree on β^*. basis for W^*

for $k = 1 \cdots m$

$$\begin{cases} \psi_2 \circ T(v_j)(g_k) = \sum_{i=1}^{m} A_{ij} \hat{w}_i(g_k) = \sum_{i=1}^{m} A_{ij} \delta_{ik} = A_{kj} \\ T^{**} \circ \psi_1(v_j)(g_k) = (\hat{v}_j \circ T^*)(g_k) = \hat{v}_j(T^*(g_k)) \end{cases}$$

$$= \hat{v}_j(g_k \circ T) = g_k \circ T(v_j)$$

$$= g_k(T(v_j)) = g_k\left(\sum_{i=1}^{m} A_{ij} w_i\right)$$

$$= \sum_{i=1}^{m} A_{ij} \cdot g_k(w_i) = \sum_{i=1}^{m} A_{ij} \delta_{ki} = A_{kj}$$

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Sec 5.1 Q17

17. Let T be the linear operator on $M_{n\times n}(R)$ defined by $T(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.
(b) Describe the eigenvectors corresponding to each eigenvalue of T.
(c) Find an ordered basis β for $M_{2\times 2}(R)$ such that $[T]_\beta$ is a diagonal matrix.
(d) Find an ordered basis β for $M_{n\times n}(R)$ such that $[T]_\beta$ is a diagonal matrix for $n > 2$.

(a) $A^t = T(A) = \lambda A \quad \therefore \quad A = (A^t)^t = T^2(A) = \lambda^2 A \quad \forall A \in M_{n\times n}$

$\therefore \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

(b) if $\lambda = 1$ $\quad A^t = T(A) = \lambda A = A \Rightarrow A$ is symmetric

if $\lambda = -1$ $\quad A^t = T(A) = \lambda A = -A \Rightarrow A$ is skew-symmetric

(c) let $\beta' = \{M_{11}, M_{12}, M_{21}, M_{22}\}$ basis for $M_{2\times 2}$

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad f_T(t) = \det([T]_{\beta'} - tI_4) = (t-1)^3(t+1)$$

eigen values are $\lambda_1 = 1$ and $\lambda_2 = -1$

For $\lambda_1 = 1$. $\quad B_1 = [T]_{\beta'} - \lambda_1 I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$B_1 x = \vec{0} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x_2 = x_3 \Leftrightarrow x \in \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_2 = x_3 \right\}$$

$\therefore E_{\lambda_1} = \{a_1 M_{11} + a_2(M_{12} + M_{21}) + a_3 M_{22} : a_i \in \mathbb{R}\}$

$= \operatorname{span}(\{M_{11}, M_{12}+M_{21}, M_{22}\})$

For $\lambda_2 = -1$, $\quad B_2 = [T]_{\beta'} - \lambda_2 I_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$B_2 x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \\ x_4 = 0 \end{cases} \Leftrightarrow x \in \{x \in \mathbb{R}^4 : x_2 + x_3 = 0,\ x_1 = x_4 = 0\}$$

$\therefore E_{\lambda_2} = \{a(M_{12} - M_{21}) : a \in \mathbb{R}\} = \operatorname{span}(\{M_{12} - M_{21}\})$

$\beta = \{M_{11}, M_{12} + M_{21}, M_{22}, M_{12} - M_{21}\}$ is an basis for $M_{2\times 2}$

s.t. $\quad [T]_\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$

(d) $\beta = \{M_{ii} : i = 1, \dots, n\} \cup \{M_{ij} + M_{ji} : 1 \le i < j \le n\} \cup \{M_{ij} - M_{ji} : 1 \le i < j \le n\}$