Homework 2 Q3

3. Sec. 1.7: Q7

Prove the following generalization of the replacement theorem. Let β be a basis for a vector space V, and let S be a linearly independent subset of V. There exists a subset S_1 of β such that $S \cup S_1$ is a basis for V.

① $\dim(V) = |\beta| < \infty$? Not necessarily.

② Even $|\beta| = \infty$, β could be a countable / uncountable set.

③ A wrong solution:

If $\dim(V) = \infty$, Let $S_1 = \phi$

while $\operatorname{span}(S \cup S_1) \subsetneq V$

select $v \in \beta$ s.t. $S \cup S_1 \cup \{v\}$ L.I. and $v \notin \operatorname{span}(S \cup S_1)$

$S_1 = S_1 \cup \{v\}$

Finally $S_1 = \{v_1, v_2, \dots\}$ and $S_1 \cup S$ is a basis for V

Let F be a field. X is a uncountable set

$V := \{ f : X \to F \ : \text{ all but finitely many } x \in X \text{ satisfy } f(x) = 0 \}$

Define Addition and Scalar multiplication for V

$$(af+g)(x) = a \cdot f(x) + g(x) \quad \forall f, g \in V \quad \forall a \in F$$

Let $\beta = \{ \delta_x : x \in X \} \subset V$ where $\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$

It's easy to prove that

- β is a uncountable set
- β is a basis for V.

Thus, we construct a vector space whose basis is a uncountable set

Homework 2 Q4

4. (Extension to Sec. 2.1: Q18) Please find **ALL** linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

- If $N(T) = R(T)$ $\quad \dim(R(T)) = \dim(N(T)) = 1 \Rightarrow T \neq T_0$

$\forall \vec{x} \in \mathbb{R}^2 \quad T^2(\vec{x}) = T(\underbrace{T(\vec{x})}_{\uparrow R(T) = N(T)}) = \vec{0} \in \mathbb{R}^2 \Rightarrow T^2 = T_0$

Claim: $N(T) = R(T) \iff T^2 = T_0$ and $T \neq T_0$

- If $T^2 = T_0$, $T \neq T_0$

$\forall \vec{y} \in R(T) \quad \exists \vec{x} \in \mathbb{R}^2$ s.t. $\vec{y} = T(\vec{x})$

$T(\vec{y}) = T(T(\vec{x})) = T^2(\vec{x}) = T_0(\vec{x}) = \vec{0}$

Thus $\vec{y} \in N(T)$ and $R(T) \subset N(T)$

Besides $\begin{cases} T \neq T_0 \Rightarrow \dim(R(T)) \geq 1 \\ T^2 = T_0 \Rightarrow \dim(R(T)) \leq 1 \end{cases} \Rightarrow \dim(R(T)) = 1$

$\Rightarrow \dim(N(T)) = 2 - 1 = 1$

Thus $R(T) = N(T)$

To find the concrete form of T, we use the fact:

T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

$\iff T(\vec{x}) = A \cdot \vec{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$

Recall that $N(T) = R(T) \iff T^2 = T_0$ and $T \neq T_0$.

$T^2(\vec{x}) = A^2 \vec{x}$ where $A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$

$T^2 = T_0 \iff A^2 = O_{2\times 2} \iff \begin{cases} a^2+bc=0 \\ ab+bd=0 \\ ac+cd=0 \\ d^2+bc=0 \end{cases}$

$T \neq T_0 \iff A \neq O_{2\times 2}$

Case 1: $b=0 \Rightarrow a=d=0$

so $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ where $c \neq 0$.

Case 2: $b \neq 0 \Rightarrow a=-d$.

Besides. $(\det A)^2 = \det A^2 = \det O_{2\times 2} = 0$

So $ad-bc = \det A = 0$, $c = -a^2/b$

Thus. $A = \begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix}$ where $b \neq 0$. $a \in \mathbb{R}$

See 2.4 Q22.

22. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define $\mathsf{T}\colon \mathsf{P}_n(F) \to \mathsf{F}^{n+1}$ by $\mathsf{T}(f) = \cancel{(f(c_0), f(c_1), \ldots, f(c_n))}$. Prove that T is an isomorphism.

$$\left(\int_0^1 f(x)\,dx,\ \int_1^2 f(x)\,dx,\ \ldots\ \int_n^{n+1} f(x)\,dx\right)$$

Proof: Recall T is an isomorphism

$\iff$ T is linear, 1-1, onto

- linear.

- 1-1. $\forall f \in N(T)$.

$$\vec{0} = T(f) = \left(\int_0^1 f(x)\,dx,\ \ldots,\ \int_n^{n+1} f(x)\,dx\right)$$

i.e $\int_k^{k+1} f(x)\,dx = 0$, $k = 0, \ldots, n$

By MVT, $\exists\, c_k \in (k, k+1)$ s.t. $\int_k^{k+1} f(x)\,dx = f(c_k)(k+1-k)$

So $\exists$ distinct $c_0, c_1, \ldots, c_n$ s.t

$f(c_0) = f(c_1) = \cdots = f(c_n) = 0$.

Since f is a polynomial of degree at most n.

we have $f(x) = 0$ $\forall x$. Thus $N(T) = \{0\}$

- onto.

$$\dim(F^{n+1}) = n+1 = \dim(P_n(F))$$
$$= \dim(N(T)) + \dim(R(T))$$
$$= \dim(R(T))$$

$R(T) \subset F^{n+1}$ Thus $R(T) = F^{n+1}$

Sec 2-4 Q25

25. Let V be a nonzero vector space over a field F, and suppose that S is a basis for V. (By the corollary to Theorem 1.13 (p. 60) in Section 1.7, every vector space has a basis). Let $\mathcal{C}(S, F)$ denote the vector space of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of vectors in S. (See Exercise 14 of Section 1.3.) Let $\Psi: \mathcal{C}(S, F) \to$ V be the function defined by

$$\Psi(f) = \sum_{s \in S, f(s) \neq 0} f(s)s.$$

Prove that Ψ is an isomorphism. Thus every nonzero vector space can be viewed as a space of functions.

Proof:

- 1-1 $\quad \forall f \in N(\psi)$. $\quad \psi(f) = 0 \in V$

  suppose $f \neq 0$ at $\{s_1 \cdots s_k\} \subset S$

  $0 = \psi(f) = f(s_1) \cdot s_1 + \cdots + f(s_k) \cdot s_k$. ✡

  since $\{s_1 \cdots s_k\}$ is L.I.

  ✡ has a unique solution $f(s_1) = \cdots = f(s_k) = 0$

  Thus $f \equiv 0$ and $N(\psi) = \{0\}$

- onto. $\quad \forall v \in V = \operatorname{span}(S)$.

  $\exists \{s_1 \cdots s_k\} \subset S$. $\quad \{a_1 \cdots a_k\} \subset F$

  s.t $\quad v = \sum_1^k a_i \cdot s_i$

  define $f(x) = \begin{cases} a_i & \text{if } x = s_i \\ 0 & \text{elsewhere.} \end{cases}$

  Then $f \in C(S, F)$ and $\psi(f) = v$.