6. Prove the following generalization of Theorem 1.9 (p. 44): Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subseteq S_2$. If S_1 is linearly independent and S_2 generates V, then there exists a basis β for V such that $S_1 \subseteq \beta \subseteq S_2$. *Hint:* Apply the maximal principle to the family of all linearly independent subsets of S_2 that contain S_1, and proceed as in the proof of Theorem 1.13.

Sec. 1.7 Q6

Zorn's Lemma.
Maximal principle

Theorem 1.9. *If a vector space* V *is generated by a finite set* S, *then some subset of* S *is a basis for* V. *Hence* V *has a finite basis.*

Proof:

dim(V) may be ∞. Extend S_1 to a basis β for V.

① Let $\mathcal{F}$ denote the family of all linearly independent subsets of S_2 that contains S_1.

② In order to show that $\mathcal{F}$ contains a maximal element, we must show that if $\mathcal{C}$ is a chain in $\mathcal{F}$, then $\exists\ \beta \in \mathcal{F}$ s.t. β contains each member of $\mathcal{C}$. We claim that β, the union of all members of $\mathcal{C}$, is the desired set.

③ Clearly, β contains each member of $\mathcal{C}$. So it suffices to prove that $\beta \in \mathcal{F}$ (i.e. β is lin. ind. subset of S_2 that contains S_1)

④ Since each member is a subset of S_2 that contains S_1, we have $S_1 \subset \beta \subset S_2$. Thus we only need to show that β is lin. ind.

⑤ let $u_1 \cdots u_n \in \beta$. $a_1 \cdots a_n \in F$ st $\sum_1^n a_i u_i = 0$

$\because u_i \in \beta$. $\exists A_i \in \mathcal{C}$ st $u_i \in A_i$.

$\because \mathcal{C}$ is a chain, $\exists A_k \in \mathcal{C}$ st $A_1, \cdots, A_n \subset A_k$.

Thus $u_1 \cdots u_n \in A_k$.

However A_k lin. ind. so. $\sum_1^n a_i u_i = 0 \Rightarrow a_1 = \cdots = a_n = 0$

Thus β lin. ind. and is a maximal lin. ind. subset of S_2

⑥ By Thm 1.12. β is a basis for $\text{span}(S_2) = V$

Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$

$$f(x) \mapsto x f(x) + f'(x)$$

find $N(T)$ and $R(T)$ and determine if T is 1-1, onto

Solution: Let $\beta = \{1, x, x^2\}$ be a basis for $P_2(\mathbb{R})$

$\gamma = \{1, x, x^2, x^3\}$

$$\begin{cases} T(1) = x \\ T(x) = x^2 + 1 \\ T(x^2) = x^3 + 2x \end{cases}$$

- $R(T) = \text{span}(T(\beta))$

$ = \text{span}(\{x, x^2+1, x^3+2x\})$ lin. ind.

$\dim(R(T)) = 3 < 4 = \dim(P_3(\mathbb{R}))$ Thus T not onto

- Recall $N(T) = \{ g \in P_2(\mathbb{R}) : T(g) = \vec{0} \}$

$= \{ a_0 + a_1 x + a_2 x^2 : T(a_0 + a_1 x + a_2 x^2) = \vec{0} \}$

$$T(a_0 + a_1 x + a_2 x^2) = a_0 \cdot T(1) + a_1 \cdot T(x) + a_2 \cdot T(x^2)$$

$$= a_0 \cdot x + a_1 (x^2+1) + a_2 (x^3 + 2x)$$

$$= a_1 + (a_0 + 2a_2) x + a_1 x^2 + a_2 x^3$$

$T(a_0 + a_1 x + a_2 x^2) = \vec{0}$ implies $\begin{cases} a_1 = 0 \\ a_0 + 2a_2 = 0 \\ a_1 = 0 \\ a_2 = 0 \end{cases} \Rightarrow a_0 = a_1 = a_2 = 0$

Thus $N(T) = \{\vec{0}\}$

Therefore T is 1-1

Sec 2.1 Q24

Definition. *Let* V *be a vector space and* W_1 *and* W_2 *be subspaces of* V *such that* $V = W_1 \oplus W_2$. *(Recall the definition of direct sum given in the exercises of Section 1.3.) A function* $T: V \to V$ *is called the* ***projection on*** W_1 ***along*** W_2 *if, for* $x = x_1 + x_2$ *with* $x_1 \in W_1$ *and* $x_2 \in W_2$, *we have* $T(x) = x_1$.

24. Let $T: R^2 \to R^2$. Include figures for each of the following parts.

(a) Find a formula for $T(a, b)$, where T represents the projection on the y-axis along the x-axis.

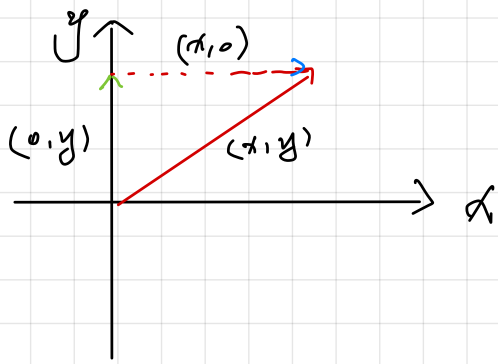
(b) Find a formula for $T(a, b)$, where T represents the projection on the y-axis along the line $L = \{(s, s): s \in R\}$.

(a) $W_1 = \{(x, 0) : x \in \mathbb{R}\}$ $\quad W_2 = \{(0, y) : y \in \mathbb{R}\}$

$\mathbb{R}^2 = W_1 \oplus W_2$

$(x, y) = (x, 0) + (0, y)$ uniquely.

$T((x, y)) = (0, y)$

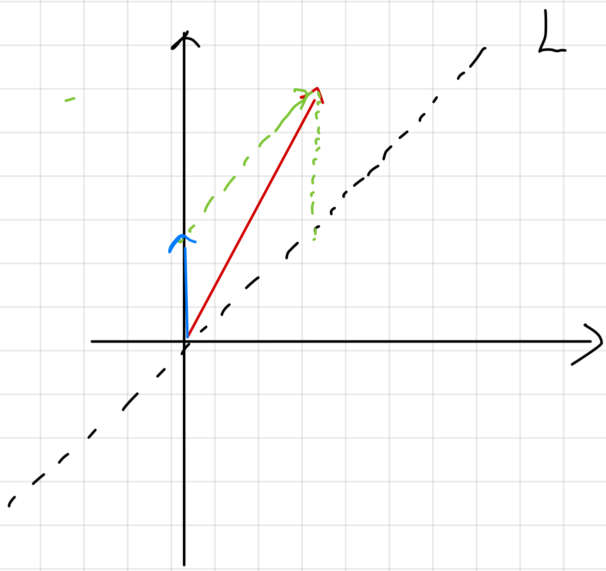


(b) $\mathbb{R}^2 = W_2 \oplus L$

$\vec{x} = \vec{x}_1 + \vec{x}_2$

let $(x, y) = \lambda(0, 1) + \beta(1, 1)$

then $\begin{cases} \lambda = y - x \\ \beta = x \end{cases}$



i.e. $\underbrace{(x, y)}_{\vec{x}} = \underbrace{(y-x)\cdot(0,1)}_{\vec{x}_1} + \underbrace{x\cdot(1,1)}_{\vec{x}_2}$

$T(\vec{x}) = \vec{x}_1 \qquad T((x, y)) = (0, y-x)$

35. Let V be a finite-dimensional vector space and $T: V \to V$ be linear.

(a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.

(b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

Be careful to say in each part where finite-dimensionality is used.

Sec 2.1 Q35

rank-nullity.

Proof:

(a) $\dim(V) < \infty$. $R(T)$ and $N(T)$ are finite-dim subspaces of V.

Therefore. $\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$

Since $V = R(T) + N(T)$.

$$\dim(V) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) \quad ①$$

However, by rank-nullity Thm.

$$\dim(V) = \dim(R(T)) + \dim(N(T)) \quad ②$$

①-②, we have. $\dim(R(T) \cap N(T)) = 0$. Thus $R(T) \cap N(T) = \{\vec{0}\}$

Therefore. $V = R(T) \oplus N(T)$

(b) It suffices to show that $V = R(T) + N(T)$

$\because R(T) \cap N(T) = \{\vec{0}\}$. $\dim(R(T) \cap N(T)) = 0$

$$\begin{aligned}\dim(R(T) + N(T)) &= \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) \\ &= \dim(R(T)) + \dim(N(T)) \\ &\overset{\text{rank-nullity.}}{=} \dim(V)\end{aligned}$$

Besides $R(T) + N(T)$ is a subspace of V.

$R(T) + N(T) = V$