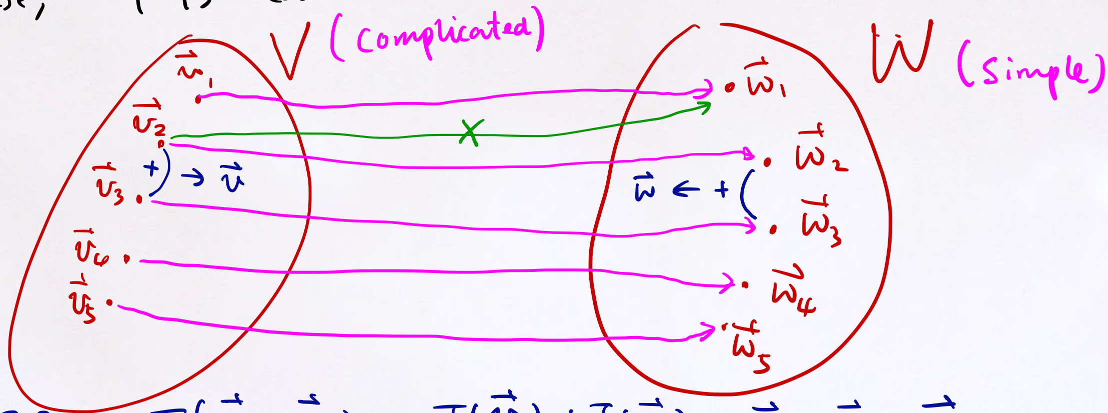
Lecture 9

Definition: Let V and W be two vector spaces.

We say V is isomorphic to W if $\exists$ an invertible linear transformation $T: V \to W$.

In this case, T is called an isomorphism from V onto W.



$$\vec{v}_2 + \vec{v}_3$$

$$\| $$

$$T(\vec{v}) = \vec{w} \;??$$

$$T(\vec{v}_2 + \vec{v}_3) = T(\vec{v}_2) + T(\vec{v}_3) = \vec{w}_2 + \vec{w}_3 = \vec{w}$$

Thm: Let V and W be finite-dimensional vector spaces.
Then: V is isomorphic to W iff $\dim(V) = \dim(W)$.

Proof: $(\Rightarrow)$ This direction follows from previous Lemma.

$(\Leftarrow)$: Suppose $\dim(V) = \dim(W) \overset{\text{def}}{=} n$ and let
$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be basis for V;
$\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be basis for W.
Then $\exists$ linear $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$
for $i = 1, 2, \ldots, n$.
By construction, T is onto and $\dim(V) = \dim(W)$.
So, T is one-to-one. $\therefore$ T is invertible.

Corollary: Let V be a vector space over F.

Then: V is isomorphic to F^n iff $\dim(V) = n$

Space of linear transformation

Prop: Let V and W be vector spaces over F.
Then: the set $\mathcal{L}(V,W)$ of all linear transformations from V to W is a vector space over F under the following operations: for linear $T, U : V \to W$,
we define: $(T+U) : V \to W$ by $(T+U)(\vec{x}) = T(\vec{x}) + U(\vec{x})$

V

and for any $a \in F$, we define $aT : V \to W$ by

$$(aT)(\vec{x}) = a\,T(\vec{x})$$

V

$T_1 : V \to W$

$T_2 : V \to W$

$\ldots$

$\mathcal{L}(V,W)$

Pf: Exercise.

Remark: If $W = V$, we write:
$\mathcal{L}(V)$ instead of $\mathcal{L}(V,V)$.

Lemma: Let V and W be finite-dim vector spaces with ordered bases β and γ respectively. Let $T, U : V \to W$ be linear.

Then: (a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma \quad \forall a \in F$

$$[aT]_\beta^\gamma = \left(\cdots \; [aT(\vec{v}_j)]_\gamma \; \cdots \right)$$

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$

<u>Pf:</u> Exercise.

Thm: Let V and W be finite-dimensional vector spaces over F. with dimension n and m respectively. Let β and γ be the ordered bases for V and W respectively.

Then: the map $\Phi : \mathcal{L}(V, W) \longrightarrow M_{m \times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Cor: $\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W) = nm$.

Proof: Φ is linear: $\Phi(T+U) = [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

$= \Phi(T) + \Phi(U)$

$\Phi(aT) = [aT]_\beta^\gamma = a[T]_\beta^\gamma$

$= a\,\Phi(T)$.

Φ is bijective:

For any $A = (A_{ij}) \in M_{m\times n}(F)$, ~~want to show that~~

$\exists!\, T: V \to W$ such that $\cancel{\Phi(T) = [T]_\beta^\gamma} = A$.

$$T(\vec{v}_j) = \sum_{i=1}^{m} A_{ij}\vec{w}_i \quad \text{for } j = 1, 2, \ldots, m$$

$$\begin{pmatrix} | \\ [T(\vec{v}_j)]_\gamma \\ | \end{pmatrix}$$

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}$

$\Downarrow$

$\therefore$ For any $A \in M_{m\times n}(F)$, $\exists!\, T: V \to W$ such that $\Phi(T) = A$. (Onto)

(1-1)

$\therefore$ Φ is bijective.

Def Let β be the ordered basis for an n-dimensional vector space V over F. The map $\phi_\beta : V \to F^n$, $\vec{x} \mapsto [x]_\beta$ is called standard representation of V with respect to β.

Prop: ϕ_β is an isomorphism.

Given vector spaces V and W of dimension n and m, with ordered bases β and γ respectively. Then, for any $T: V \to W$ (linear), we have:

$$\begin{tikzcd}
\vec{v} \in V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \ni T(\vec{v}) := \vec{w} \arrow[d, "\phi_\gamma"] \\
[\vec{v}]_\beta \in F^n \arrow[r, "L_A"] & F^m \quad [\vec{w}]_\gamma = [T(\vec{v})]_\gamma
\end{tikzcd}$$

where $A = [T]_\beta^\gamma$

$$\Rightarrow \phi_\gamma \circ T(\vec{v}) = L_A \circ \phi_\beta(\vec{v})$$

$$\Leftrightarrow [T(\vec{v})]_\gamma = [T]_\beta^\gamma [\vec{v}]_\beta$$

Dual Space

Let V be a vector space over F.

Definition: A linear functional on V is a linear map $f : V \to F$.

Remark: A linear functional belongs to $\mathcal{L}(V, F)$.

Definition: The dual space, denoted by V^*, is the space of all linear functional on V. That is, $V^* = \mathcal{L}(V, F)$.

Next time: Suppose V is finite-dimensional. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. For each $i = 1, 2, \ldots, n$, define a linear functional f_i by setting:

$$f_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then: $\{f_1, f_2, \ldots, f_n\}$ is a basis of V^*, called the dual basis of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. $\therefore \dim(V) = \dim(V^*)$