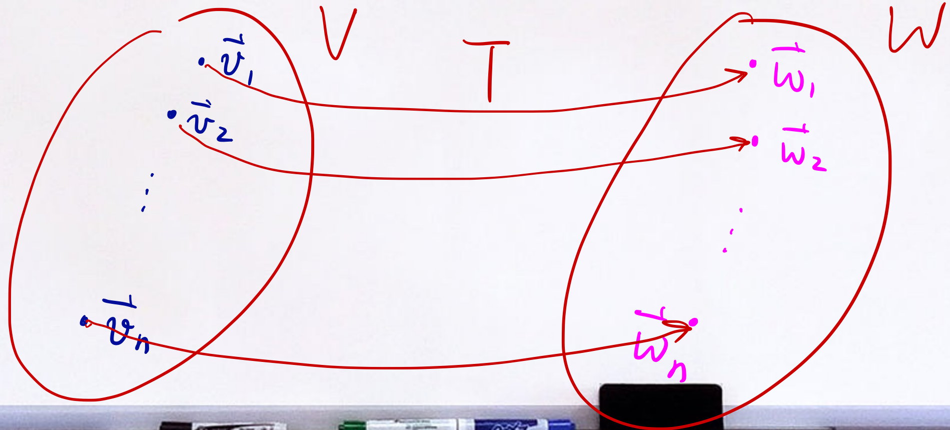
Lecture 6:

Thm: Let V and W be vector spaces. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. Then, given any $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$. $\exists$ a unique linear transformation $T : V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$



Proof: For $\vec{x} \in V$, $\exists!\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$.

We define $T : V \to W$ by : $T(\vec{x}) = \sum_{i=1}^{n} a_i \vec{w}_i \in W$

- T is linear.: For $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \in V$, $\vec{y} = \sum_{i=1}^{n} b_i \vec{v}_i \in V$

and $c \in F$,

we have:
$$
\begin{aligned}
T(c\vec{x} + \vec{y}) &= T\left(\sum_{i=1}^{n} (c a_i + b_i) \vec{v}_i \right) \\
&= \sum_{i=1}^{n} (c a_i + b_i) \vec{w}_i \\
&= c \underbrace{\left(\sum_{i=1}^{n} a_i \vec{w}_i \right)}_{= T(\vec{x})} + \underbrace{\left(\sum_{i=1}^{n} b_i \vec{w}_i \right)}_{= T(\vec{y})}
\end{aligned}
$$

- By definition, $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$

- T is unique: Suppose $U : V \to W$ is linear s.t.

$$U(\vec{v}_i) = \vec{w}_i \quad \text{for } \forall i.$$

For any $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \in V$, we have:

$$U(\vec{x}) = \sum_{i=1}^{n} a_i \underbrace{U(\vec{v}_i)}^{\vec{w}_i} = \sum_{i=1}^{n} a_i \vec{w}_i = T(\vec{x}).$$

$$\therefore \quad U = T.$$

Corollary: Let V be a vector space with a finite basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

Then any linear transformation from V to another vector space W is completely determined by its values on β.

(That is, if U and T are linear transformations from V to W s.t. $U(\vec{v}_i) = T(\vec{v}_i)$, then $U = T$)

Matrix representation

Notation: An ordered basis for a finite-dimensional vector space V is a basis for V endowed with a specific order.

(e.g. $\mathbb{R}^2$ $\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{= \beta_1} \neq \underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}_{= \beta_2}$ as ordered basis)

Definition: Let V be a finite-dimensional vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an ordered basis for V.

Then, $\forall \vec{x} \in V$, $\exists !\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i$.

The coordinate vector of $\vec{x}$ relative to β, denoted as $[\vec{x}]_\beta$, is the column vector (F^n)

$$[\vec{x}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

Remark: Define a map $V \to F^n$. This map is linear

$$\vec{x} \mapsto [\vec{x}]_\beta$$

(HW. $[\underbrace{a\vec{x}+\vec{y}}_{\in V}]_\beta = a[\vec{x}]_\beta + [\vec{y}]_\beta$)

Now, suppose V and W are finite-dimensional vector spaces with ordered bases $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ (for V) and $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ (for W) respectively.

Let $T: V \to W$ be a linear transformation.

Then for each $1 \le j \le n$, $\exists\, a_{ij} \in F$ $1 \le i \le m$ such that

$$\underbrace{T(\vec{v}_j)}_{\in W} = \sum_{i=1}^{m} a_{ij} \vec{w}_i \quad \text{for} \quad 1 \le j \le n.$$

<u>Definition:</u> With this notation as above, we call the matrix

$$A \overset{\text{def}}{:=} (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

the **matrix representation of T in the ordered bases β and γ**, and denoted it as $A = [T]_\beta^\gamma$.

If $V = W$ and $\beta = \gamma$, then we simply write $\cancel{[T]_\beta^\beta}$ $[T]_\beta$

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

W

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij}\,\vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

$$\in W$$

$$A = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}} \end{pmatrix}$$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & [T(\vec{v}_n)]_\gamma \end{array}$$

$$T(\vec{v}_1) = \sum_{i=1}^{m} a_{i1}\,\vec{w}_i$$

$$\in W$$

$$\Downarrow$$

$$[T(\vec{v}_1)]_\gamma = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \in F^n$$

$$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \quad \text{for } V$$

$$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\} \quad \text{for } W$$

$$[T]_\beta^\gamma = \begin{array}{cc} m & \underbrace{\left[\begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix}\right.}_{n} \end{array}$$

$$\in$$

$$M_{m \times n}$$