Lecture 5:

Proposition: Let $T: V \to W$ be a linear transformation.

Then: $N(T)$ and $R(T)$ are subspaces of V and W respectively.

Proof: $\because$ $T(\vec{0}_V) = \vec{0}_W$

$\therefore$ $\vec{0}_V \in N(T)$ and $\vec{0}_W \in R(T)$

Let $\vec{x}$ and $\vec{y} \in N(T)$ and $a \in F$. Then:

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0}_W + \vec{0}_W = \vec{0}_W \quad \text{and}$$

$$T(a\vec{x}) = aT(\vec{x}) = a\vec{0}_W = \vec{0}_W$$

$\therefore$ $\vec{x} + \vec{y} \in N(T)$ and $a\vec{x} \in N(T)$

$\therefore$ $N(T)$ is a subspace of V.

Now, let $\vec{u}, \vec{v} \in R(T)$ and $a \in F$.

Then: $\exists\, \vec{x}, \vec{y} \in V$ such that $\vec{u} = T(\vec{x})$ and $\vec{v} = T(\vec{y})$

So, $T(\underbrace{\vec{x}+\vec{y}}_{\in V}) = T(\vec{x}) + T(\vec{y}) = \vec{u} + \vec{v} \Rightarrow \vec{u} + \vec{v} \in R(T)$

$T(\underbrace{a\vec{x}}_{\in V}) = a\,T(\vec{x}) = a\vec{u} \Rightarrow a\vec{u} \in R(T)$

$\therefore$ $R(T)$ is a subspace of W.

Remark: $T: V \to W$ is onto iff $R(T) = W$

(follows from the def)

Proposition: A linear transformation $T: V \to W$ is one-to-one iff $N(T) = \{\vec{0}\}$.

Pf: (Recap: One-to-one $\Leftrightarrow$ " $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$ "

($\Rightarrow$) If T is one-to-one, then: for any $\vec{x} \in N(T)$,

we have $T(\vec{x}) = \vec{0}_W = T(\vec{0}_V)$

$\Rightarrow \vec{x} = \vec{0}_V$

This implies $N(T) = \{\vec{0}_V\}$.

$(\Leftarrow)$ Suppose $N(T) = \{\vec{0}_V\}$

Let $\vec{x}, \vec{y} \in V$ such that $T(\vec{x}) = T(\vec{y})$.

Then: $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$

This implies $\vec{x} - \vec{y} \in N(T) = \{\vec{0}_V\}$

$\therefore \vec{x} - \vec{y} = \vec{0}_V$ or $\vec{x} = \vec{y}$.

$\therefore T$ is 1-1.

<u>Definition</u>: Let $T: V \to W$ be a linear transformation.

If $N(T)$ and $R(T)$ are finite-dimensional, we define:

- Nullity is denoted as Nullity(T) is the dimension of $N(T)$.
- Rank is denoted as Rank(T) is the dimension of $R(T)$.

<u>Lemma</u>: Let $T: V \to W$ be a linear transformation. If $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for V, then:

$$R(T) = \text{Span}(T(\beta)) \overset{\text{def}}{=} \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)\}$$

Proof: $\because$ $T(\vec{v}_j) \in R(T)$ for $j = 1, 2, \dots, n$

and $R(T)$ is subspace.

$$\therefore \ \mathrm{Span}\{\underset{\substack{\uparrow \\ R(T)}}{T(\vec{v}_1)}, \underset{\substack{\uparrow \\ R(T)}}{T(\vec{v}_2)}, \dots, \underset{\substack{\uparrow \\ R(T)}}{T(\vec{v}_n)}\} \subset R(T)$$

Conversely, let $\underset{\substack{\parallel \\ T(\vec{x})}}{\vec{w}} \in R(T)$ where $\vec{x} \in V$.

Then: $\exists \ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{j=1}^{n} a_j \vec{v}_j$.

$$\text{So, } \vec{w} = T(\vec{x}) = T\left(\sum_{j=1}^{n} a_j \vec{v}_j\right) = \sum_{j=1}^{n} a_j T(\vec{v}_j) \in \mathrm{Span}\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$$

$\therefore \ R(T) \subset \mathrm{Span}(T(\beta))$ $\quad \therefore \ R(T) = \mathrm{Span}(T(\beta))$

Example: $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(f) = \begin{pmatrix} f(0) & f(1) \\ 0 & f'(0) \end{pmatrix}$$

We have: $R(T) = \operatorname{span}\{T(1),\ T(x),\ T(x^2)\}$

$$= \operatorname{span}\underbrace{\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}}_{\text{Lin. indep.}}$$

$\Rightarrow \quad \operatorname{Rank}(T) = 3$

Theorem: (Rank - Nullity Theorem)

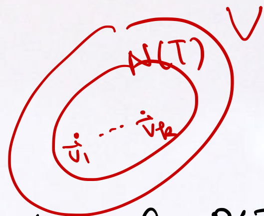
Let V and W be vector spaces s.t. V is finite-dimensional.

Then for any linear transformation $T: V \to W$, we have:

$$\text{nullity}(T) + \text{Rank}(T) = \dim(V)$$

Proof: Let $n = \dim(V)$ and $k = \dim(N(T)) \leq n$

Choose a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ for $N(T)$ and extend it to a basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.



Claim: $S = \{T(\vec{v}_{k+1}), T(\vec{v}_{k+2}), \ldots, T(\vec{v}_n)\}$ is a basis for $R(T)$.

① $$R(T) = \text{span}\{\underset{\substack{\| \\ 0}}{T(\vec{v}_1)}, T(\vec{v}_2), \ldots, \underset{\substack{\| \\ 0}}{T(\vec{v}_k)}, T(\vec{v}_n)\}$$

$$= \text{span}\{\underbrace{T(\vec{v}_{k+1}), \ldots, T(\vec{v}_n)}_{S}\} = \text{span}(S)$$

② Now suppose $\exists\, b_{k+1}, b_{k+2}, \ldots, b_n \in F$ s.t.

$$\sum_{i=k+1}^{n} b_i T(\vec{v}_i) = \vec{0}.$$

Then, by linearity, we have: $T\left(\sum_{i=k+1}^{n} b_i \vec{v}_i\right) = \vec{0}$

$$\Rightarrow \sum_{i=k+1}^{n} b_i \vec{v}_i \in N(T)$$

$\therefore \displaystyle\sum_{i=k+1}^{n} b_i \vec{v}_i = \sum_{i=1}^{k} c_i \vec{v}_i$ for some $c_1, \ldots, c_k \in F$.

But then: $\displaystyle\sum_{i=1}^{k} (-c_i) \vec{v}_i + \sum_{i=k+1}^{n} b_i \vec{v}_i = \vec{0}$

$\therefore$ $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V and so it is lin. ind.

$\therefore$ $(-c_i) = 0$ for $i = 1, 2, \ldots, k$

$b_i = 0$ for $i = k+1, k+2, \ldots, n$

$\therefore$ S is lin. ind.

$\therefore$ S is basis for $R(T)$.

$$\therefore \quad \text{Nullify}(T) + \text{Rank}(T)$$

$$= k + (n-k)$$

$$= n = \dim(V)$$

Example: Consider $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by:

$$T(f(x)) \overset{\text{def}}{:=} \underline{2f'(x)} + \underline{\int_0^x 3f(t)\,dt}$$

We have $R(T) = \text{span}\{T(1), T(x), T(x^2)\}$

$$= \text{span}\{\underbrace{3x,\ 2+\tfrac{3}{2}x^2,\ 4x+x^3}_{\text{Linear independent}}\}$$

$$\dim(R(T)) = \text{rank}(T) = 3$$

$$\cancelto{3}{\text{Rank}(T)} + \text{Nullity}(T) = \dim(\cancelto{3}{P_2(\mathbb{R})})$$

$$\Rightarrow \text{Nullity}(T) = 0 \Rightarrow N(T) = \{\vec{0}\}$$

$$\Rightarrow T \text{ is one-to-one.}$$

Thm: Let V and W be vector spaces of equal finite-dimensions. Let $T: V \to W$ be a linear transformation.
Then, the following are equivalent:

(a) T is one-to-one

(b) T is onto

(c) $\text{Rank}(T) = \dim(V)$

$\dim(R(T)) \leq \dim(W)$

Proof: T is one-to-one

$\iff$ $\text{Nullity}(T) = 0$ (by previous proposition)

$\iff$ $\text{Rank}(T) + \cancelto{0}{\text{Nullity}(T)} = \dim(V)$

$\iff$ $\underset{\dim(R(T))}{\underset{\|}{\text{Rank}(T)}} = \dim(W)$ $\iff$ $R(T) = W$

$\iff$ T is onto

Example: Show that $\forall\, q(x) \in P(\mathbb{R})$, $\exists\, p(x) \in P(\mathbb{R})$ such that

$\forall$: for all $\qquad$ $\exists$: there exists

$$[(x^2+5x+7)\,p(x)]'' = q(x).$$

$\Updownarrow$

(T is onto)

Consider $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by:

$$T(p(x)) = [(x^2+5x+7)\,p(x)]''$$

(Exercise: T is linear)

(~~Need to check $N(T)=\{\vec{0}\}$ or $\text{Nullity}(T) = 0$~~)

because $\dim(P(\mathbb{R})) = \infty$

Idea: Restrict T to $P_n(\mathbb{R})$: Define: $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$

such that $T(p(x)) = [(x^2+5x+7)\,p(x)]''$

Remain to show $\text{Nullity}(T) = 0$. (Exercise)