Lecture 22:

Cor: If $F = \mathbb{C}$, then T is normal iff $T^* = g(T)$ for some polynomial g.

Pf: ($\Rightarrow$) Suppose T is normal. Let $T = \lambda_1 T_1 + \ldots + \lambda_k T_k$ be spectral decomposition of T.

Then:
$$T^* = \overline{\lambda_1} T_1^* + \overline{\lambda_2} T_2^* + \ldots + \overline{\lambda_k} T_k^*$$
$$= \overline{\lambda_1} T_1 + \overline{\lambda_2} T_2 + \ldots + \overline{\lambda_k} T_k$$

By Lagrange interpolation, $\exists$ a polynomial g s.t. $g(\lambda_i) = \overline{\lambda_i}$ $\forall i = 1, 2, \ldots k$

Then:
$$g(T) = g(\lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_k T_k)$$
$$= g(\lambda_1) T_1 + g(\lambda_2) T_2 + \ldots + g(\lambda_k) T_k \quad \text{(Check)}$$
$$= \overline{\lambda_1} T_1 + \overline{\lambda_2} T_2 + \ldots + \overline{\lambda_k} T_k = T^*$$

$g = x^2 + x$

$(\lambda_1 T_1 + \lambda_2 T_2)^2 + (\lambda_1 T_1 + \lambda_2 T_2)$

$\lambda_1^2 \underset{T_1}{\underset{\parallel}{T_1^2}} + \lambda_2^2 T_2^2 = T_2$

$+ \lambda_1 \lambda_2 \cancel{T_1 T_2}^{0} + \lambda_1 \lambda_2 \cancel{T_2 T_1}^{0}$

($\Leftarrow$) If $T^* = g(T)$, then : $T^*T = g(T)\,T = T\,g(T)$

$$= TT^*$$

$\therefore$ T is normal.

Jordan Canonical Form

Recall: Let $T: V \to V$ lin. operator on a fin-dim V (over F)

$T: V \to V$ is diagonalizable $\iff$

1. Char poly splits.
2. $\dim(E_{\lambda_i}) = m_i$ ← alg. multiplicity for all eigenvalues λ_i

(In general, $\dim(E_{\lambda_i}) \leq m_i$)

Remark: "Diagonalizable" (as a linear transf) $\iff$ eigenspaces are BIG enough

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

$$B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

$$K = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

Theorem: Any $A \in M_{n\times n}(\mathbb{C})$ is similar to a matrix of the following form:

$$J = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & & 0\\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1\end{matrix}} & & & 0 \\ & \boxed{\begin{matrix}\lambda_2 & 1 & & 0\\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix}\lambda_N & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_N\end{matrix}} \end{pmatrix}$$

(Jordan Canonical Form of A)

$\lambda_1, \lambda_2, \ldots, \lambda_N$ are eigenvalues of A
(not necessarily distinct)

$$\begin{pmatrix} \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} & & \\ & \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} & \\ & & \begin{pmatrix} 3 & 1 \\ & 3 \end{pmatrix} \end{pmatrix}$$

Given $T: V \to V$, V is fin-dim.

Find a basis β of V $\ni$ $[T]_\beta = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$

(A_i = block square matrix)

$A_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ where λ is an eigenvalue of T.

Remark:

- $[T]_\beta$ is called the Jordan Canonical form of T
- A_i is called a Jordan block corresponding to λ
- β is called the Jordan canonical basis.

Remark: Jordan canonical consists of blocks in this form:

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \lambda \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_{k \times k}(\mathbb{C})$$

It is called Jordan block of size k with eigenvalue λ.

Prop: (1) A has only 1 eigenvalue λ (multiplicity is k)

(2) $\dim(E_\lambda) = 1$ ($\Rightarrow$ A is not diagonalizable if $k \neq 1$)

(3). The smallest positive integer p s.t.

$(A - \lambda I)^p = 0$ is equal to the dimension k.

($\Rightarrow$ $N((A - \lambda I)^p) = \mathbb{C}^k$)

(4) If $\{\vec{e}_1, \dots, \vec{e}_k\}$ is the standard basis for $\mathbb{C}^k$,

then $(A - \lambda I)^i \vec{e}_i = 0$ for each $i = 1, 2, \dots, k$.

e.g. $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Then: $(A - \lambda I) = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

$$(A - \lambda I)^3 = 0$$

<u>Definition:</u> Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{C})$. $\vec{x} \in \mathbb{C}^n$ is a generalized eigenvector of A corresponding to the eigenvalue λ if (i) $\vec{x} \neq \vec{0}$

and (ii) $(A - \lambda I)^p \vec{x} = 0$ for some positive integer p.

We denote the generalized eigenspace by:

$$K_\lambda = \{ \vec{x} \in \mathbb{C}^n : (A - \lambda I)^p \vec{x} = \vec{0} \text{ for some } p \geq 1 \}$$

Main Theorem: (Jordan Decomposition Theorem)

Let $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (distinct) with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Then:

(1) $\dim(K_{\lambda_i}) = m_i$

(2) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$

(3) Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{\ell,i}$ where every $\gamma_{m,i}$ is a cycle:

$$\gamma_{m,i} = \{ \underbrace{(A-\lambda_i I)^{p-1}\vec{x}}_{\text{eigenvector}}, (A-\lambda_i I)^{p-2}\vec{x}, \quad \ldots, (A-\lambda_i I)\vec{x}, \vec{x} \}$$

gives rise to

$$\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & \ddots \\ 0 & & & \lambda_i \end{pmatrix}$$