Lecture 19:

$W \subset V$ subspace

$$W^{\perp} = \{ \vec{x} \in V : \langle \vec{x}, \vec{w} \rangle = 0 \text{ for } \forall \vec{w} \in W \}$$

Proposition: Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then:

(a) S can extended to an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

(b) If $W = \text{span}(S)$, then $S_1 = \{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is an orthonormal basis for $W^\perp$

(c) If W is any subspace of V, then:

$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Proof: (a) We first extend S to a basis:

$$\{\overbrace{\underbrace{\vec{v}_1, \dots, \vec{v}_k}_{L.I.}}^{S}, \vec{w}_{k+1}, \dots, \vec{w}_n\} \text{ for } V.$$

Then, we apply the G-S process to this basis.

$\because$ S is orthonormal, $\therefore$ $\vec{v}_1, \dots, \vec{v}_k$ remains the same during the G-S process.

So, this process gives an orthonormal basis for V of the form $\{\underbrace{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k}_{\text{unchanged}}, \underbrace{\vec{v}_{k+1}, \dots, \vec{v}_n}_{\text{new}}\}$

(c) For any W, choose an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for W and extend it to an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for V.

Then:

$$\dim(V) = n = k + (n-k)$$
$$= \dim(W) + \dim(W^\perp)$$

For $\mathbb{R}^n$:

$$A\vec{x} \cdot \vec{y} = (A\vec{x})^T \vec{y} = \vec{x}^T (A^T \vec{y}) = \vec{x} \cdot A^T \vec{y}$$

$??$

Adjoint of a linear operator

Prop: Let V be a finite-dim. inner product space over F.
Then for any linear transformation $g : V \to F$ (linear functional),
$\exists ! \, \vec{y} \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Proof: Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for V.

Set : $\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)} \, \vec{v}_i$.

We have: $\langle \vec{v}_j, \vec{y} \rangle = \sum_{i=1}^{n} g(\vec{v}_i) \langle \vec{v}_j, \vec{v}_i \rangle = g(\vec{v}_j)$ for $\forall j$

$\Rightarrow \quad g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$

If $\exists \vec{y}' \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for $\forall \vec{x}$.

then, $\langle \vec{x}, \vec{y} \rangle = g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for $\forall \vec{x}$

$$\Rightarrow \quad \vec{y} = \vec{y}'.$$

Theorem: Let V be a finite-dim inner product space. Let T be a linear operator on V. Then: $\exists!$ linear operator $T^*: V \to V$ such that: $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$.

T^* is called the adjoint of T.

Proof: Given any $\vec{y} \in V$, the map $g_{\vec{y}}: V \to F$ defined by:

$$g_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y} \rangle$$

is linear ($\because$ $\langle \cdot, \cdot \rangle$ is linear in the 1st argument) + T is linear

By the previous proposition, $\exists!\ \vec{y}\,' \in V$ such that $\underbrace{\langle T(\vec{x}), \vec{y} \rangle}_{= g_{\vec{y}}(\vec{x})} = \langle \vec{x}, \vec{y}\,' \rangle$ for all $\vec{x} \in V$.

Now, we uniquely define: $T^*: V \to V$ by $T^*(\vec{y}) = \vec{y}\,'$.

To see that T^* is linear, let $\vec{y}_1, \vec{y}_2 \in V$ and $c \in F$.

Then $\forall \vec{x} \in V$, we have:

$$
\begin{aligned}
\langle \vec{x}, T^*(c\vec{y}_1 + \vec{y}_2) \rangle &= \langle T(\vec{x}), c\vec{y}_1 + \vec{y}_2 \rangle \\
&= \bar{c} \langle T(\vec{x}), \vec{y}_1 \rangle + \langle T(\vec{x}), \vec{y}_2 \rangle \\
&= \bar{c} \langle \vec{x}, T^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle \\
&= \langle \vec{x}, c\,T^*(\vec{y}_1) + T^*(\vec{y}_2) \rangle
\end{aligned}
$$

$$\Rightarrow \quad T^*(c\vec{y}_1 + \vec{y}_2) = c\,T^*(\vec{y}_1) + T^*(\vec{y}_2)$$

Remark:

$$\langle \vec{x}, T(\vec{y}) \rangle = \overline{\langle T(\vec{y}), \vec{x} \rangle} = \overline{\langle \vec{y}, T^*(\vec{x}) \rangle} = \langle T^*(\vec{x}), \vec{y} \rangle$$

Proposition: Let V be a finite-dim inner product space and let β be an orthonormal basis for V. Then $\forall T: V \to V$, we have:

$$[T^*]_\beta = ([T]_\beta)^* \leftarrow \text{conjugate tranpose}$$

$$\left(A^* = (\overline{A})^T \right)$$

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$ and $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

$$B_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i \rangle = \langle \vec{v}_j, T(\vec{v}_i) \rangle = \overline{\langle T(\vec{v}_i), \vec{v}_j \rangle}$$

$$= \overline{A_{ji}}$$

Corollary: Let A be an $n \times n$ matrix. Then: $L_{A^*} = (L_A)^*$

conjugate transpose

adjoint

Pf: The standard basis β for F^n is orthonormal.

Then: $[L_A]_\beta = A$.

$$\therefore [(L_A)^*]_\beta = ([L_A]_\beta)^* = A^* = [L_{A^*}]_\beta \Rightarrow (L_A)^* = L_{A^*}$$

Let $A = [T]_\beta$, $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ = o.n. basis.

Then: $T(\vec{v}_j) = \sum_{i=1}^{n} A_{ij} \vec{v}_i$

$$= \sum_{i=1}^{n} \langle T(\vec{v}_j), \vec{v}_i \rangle \vec{v}_i$$

$$\therefore \quad A_{ij} = T(\vec{v}_j)$$

Proposition: Let V be an inner product space. Let $T, U = V \to V$.

Then:

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\,T^* \quad \forall c \in F$

(c) $(TU)^* = U^* T^*$

(d) $(T^*)^* = T$

(e) $I^* = I$

Proof: $\forall \vec{x}, \vec{y} \in V$

(a) $\langle \vec{x}, (T+U)^*(\vec{y}) \rangle = \langle (T+U)(\vec{x}), \vec{y} \rangle = \langle T(\vec{x}), \vec{y} \rangle + \langle U(\vec{x}), \vec{y} \rangle$

$= \langle \vec{x}, T^*(\vec{y}) \rangle + \langle \vec{x}, U^*(\vec{y}) \rangle$

$= \langle \vec{x}, (T^* + U^*)(\vec{y}) \rangle$

$\Rightarrow (T+U)^* = T^* + U^*.$

(b) $\langle \vec{x}, (cT)^*(\vec{y}) \rangle = \langle cT(\vec{x}), \vec{y} \rangle$

$= c \langle T(\vec{x}), \vec{y} \rangle$

$= c \langle \vec{x}, T^*(\vec{y}) \rangle = \langle \vec{x}, \bar{c} T^*(\vec{y}) \rangle$

$\therefore (cT)^* = \bar{c} T^*$

(c) $\langle \vec{x}, (TU)^*(\vec{y}) \rangle = \langle T(U(\vec{x})), \vec{y} \rangle$

$= \langle U(\vec{x}), T^* \vec{y} \rangle$

$= \langle \vec{x}, U^* T^* \vec{y} \rangle$

$\Rightarrow (TU)^* = U^* T^*.$

(d) $\langle \vec{x}, T(\vec{y}) \rangle = \langle T^*(\vec{x}), \vec{y} \rangle = \langle \vec{x}, (T^*)^*(\vec{y}) \rangle$

$\Rightarrow \quad T = T^{**}$.

(e). follows from the definition.

$$\langle \vec{x}, I(\vec{y}) \rangle = \langle I(\vec{x}), \vec{y} \rangle$$

$$\| $$

$$\langle \vec{x}, \vec{y} \rangle$$

<u>Remark:</u> Let A and B be $n \times n$ matrices. Then:

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c} A^*$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $I^* = I$.

Lemma: Let $T : V \to V$ be a linear operator on a finite-dim inner product space V. If T has an eigenvector, then so does T^*.

Pf: Suppose $\vec{v} \in V \setminus \{\vec{0}\}$ is an eigenvector of T with eigenvalue λ.

Then: $\forall \vec{x} \in V$, we have:

$$0 = \langle \vec{0}, \vec{x} \rangle = \langle (T-\lambda I)(\vec{v}), \vec{x} \rangle = \langle \vec{v}, \underbrace{\overset{T^* - \bar{\lambda} I}{\overset{\parallel}{(T-\lambda I)^*}}(\vec{x})}_{\in R(T^* - \bar{\lambda} I)} \rangle$$

$\Rightarrow \vec{v} \in R(T^* - \bar{\lambda} I)^{\perp}$. So, $\dim(R(T^* - \bar{\lambda} I)) < \dim(V)$.

$(\dim(W) + \dim(W^{\perp}) = \dim(V))$

$\Rightarrow \dim(N(T^* - \bar{\lambda} I)) > 0$ $\therefore$ T^* has an eigenvector with eigenvalue $\bar{\lambda}$.

Thm (Schur) Let T be a lin. operator on a finite-dim inner product space. Suppose the char. poly of T splits.

Then: $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is upper triangular.

Pf: We prove by induction on $n = \dim(V)$.

The $n=1$ case is obvious.

Assume the statement holds for lin. operators defined on $(n-1)$-dim inner product space, whose char. poly splits.

By lemma, T^* has a unit eigenvector $\vec{z}$.

Let $W \overset{\text{def}}{:=} \operatorname{span}\{\vec{z}\}$ and suppose $T^*(\vec{z}) = \lambda \vec{z}$.

Claim: $W^\perp$ is T-invariant.

Pf: Let $\vec{y} \in W^\perp$ and $\vec{x} = c\vec{z} \in W$. Then:

$$\begin{aligned}
\langle T(\vec{y}), \vec{x} \rangle = \langle T(\vec{y}), c\vec{z} \rangle &= \langle \vec{y}, cT^*(\vec{z}) \rangle \\
&= \langle \vec{y}, c\lambda \vec{z} \rangle \\
&= \bar{c}\bar{\lambda} \langle \underset{\substack{\uparrow \\ W^\perp}}{\vec{y}}, \underset{\substack{\uparrow \\ W}}{\vec{z}} \rangle = 0
\end{aligned}$$

$\therefore$ $T(\vec{y}) \in W^\perp$.

Now, $f_{T_{W^\perp}}(t) \mid f_T(t) \Rightarrow f_{T_{W^\perp}}(t)$ splits. ①

Also, $\dim(W^\perp) = n-1$ ②

$\therefore$ Induction hypothesis gives an orthonormal basis γ for $W^\perp$ s.t. $[T_{W^\perp}]_\gamma$ is upper triangular.

Then: $\beta \overset{def}{=} \underset{W^\perp}{\underset{\cap}{\gamma}} \cup \{ \underset{W}{\underset{\cap}{\vec{z}}} \}$ is orthonormal basis s.t.

$$[T]_\beta = \begin{pmatrix} [T_{W^\perp}]_\gamma & \begin{matrix} * \\ * \\ * \\ * \\ \vdots \end{matrix} \\ 0 \cdots\cdots 0 & * \end{pmatrix}$$ is upper triangular