Lecture 16: Recall:

Tensor Product Space

Definition: A map $B : X \times Y \to Z$ (X, Y, Z are vector spaces over F) is a bilinear map if:

$$B(a\vec{u} + b\vec{v}, c\vec{s} + d\vec{t}) = ac\, B(\vec{u}, \vec{s}) + ad\, B(\vec{u}, t) + bc\, B(\vec{v}, \vec{s}) + bd\, B(\vec{v}, t)$$

where $\vec{u}, \vec{v} \in X$, $\vec{s}, t \in Y$, $a, b, c, d \in F$.

Definition: Let V_1, V_2 be vector spaces over F. A pair (Y, μ), where Y is a vector space over F and $\mu : V_1 \times V_2 \to F$ is a bi-linear map, is called the tensor product of V_1 and V_2 if the following condition holds:

(*) whenever β_1 is a basis for V_1 and β_2 is a basis for V_2, then:

$$\mu(\beta_1 \times \beta_2) \overset{\text{def}}{:=} \{ \mu(\vec{x}_1, \vec{x}_2) : \vec{x}_1 \in \beta_1, \vec{x}_2 \in \beta_2 \}$$ is a basis for Y.

Remark: We write $V_1 \otimes V_2$ for Y.

We write $\vec{x}_1 \otimes \vec{x}_2$ for $\mu(\vec{x}_1, \vec{x}_2)$.

Theorem: Let Y be a vector space and $\mu : V_1 \times V_2 \to Y$ be a bi-linear map. Suppose there exist bases γ_1 for V_1 and γ_2 for V_2 such that $\mu(\gamma_1 \times \gamma_2)$ is a basis for Y. Then $(*)$ holds for any choice of basis.

Proof: Let β_1 and β_2 be bases for V_1 and V_2 respectively.

① $\mu(\beta_1 \times \beta_2)$ spans Y. :

$\because$ $\mu(\gamma_1 \times \gamma_2)$ spans Y, we have:

$$y = \sum_{j,k} a_{jk}\, \mu(\vec{z}_{1j}, \vec{z}_{2k}) \quad , \quad \vec{z}_{1j} \in \gamma_1 \ , \ \vec{z}_{2k} \in \gamma_2 .$$

Write $\vec{z}_{1j} = \sum_l b_{jl} \vec{x}_{1l}$ where $\vec{x}_{1l} \in \beta_1$ and

$\vec{z}_{2k} = \sum_m c_{km} \vec{x}_{2m}$ where $\vec{x}_{2m} \in \beta_2$.

$$\therefore \quad \vec{y} = \sum_{j,k} a_{jk} \, \mu\Big(\sum_l b_{jl} \vec{x}_{1l} , \sum_m c_{km} \vec{x}_{2m} \Big)$$

$$= \sum_{j,k,l,m} a_{jk} \, b_{jl} \, c_{km} \, \mu(\vec{x}_{1l}, \vec{x}_{2m})$$

$\therefore \quad y \in \mathrm{Span}\big(\mu(\beta_1 \times \beta_2)\big)$.

(2) $\mu(\beta_1 \times \beta_2)$ is linearly independent.

$\because \quad \underbrace{|\mu(\beta_1 \times \beta_2)|}_{\text{Spanning set}} = \underbrace{|\mu(\gamma_1 \times \gamma_2)|}_{\text{Basis}}$ $\quad \therefore \quad \mu(\beta_1 \times \beta_2)$ is L.I.

For infinite dimensions, we can prove:

$\mu(\beta_1 \times \beta_2)$ is L.I. iff $\mu(\gamma_1 \times \gamma_2)$ is L.I.

Suppose $\mu(\gamma_1 \times \gamma_2)$ is L.I.

Consider $\displaystyle\sum_{l,m} d_{lm}\, \mu(\vec{x}_{1l}, \vec{x}_{2m}) = 0$ where $\vec{x}_{1l} \in \beta_1$, $\vec{x}_{2l} \in \beta_2$

Write $\displaystyle\vec{x}_{1l} = \sum_j e_{lj}\, \vec{z}_{1j}$ where $\vec{z}_{1j} \in \gamma_1$ and

$\displaystyle\vec{x}_{2m} = \sum_k f_{mk}\, \vec{z}_{2k}$ where $\vec{z}_{2k} \in \gamma_2$.

$$\begin{tikzcd} V \arrow[r, "(e_{lj})"] & V \\ \gamma_1 & \beta_1 \\ V & V \arrow[l, "(b_{jl})"] \end{tikzcd}$$

Also, write $\displaystyle\vec{z}_{1j} = \sum_l b_{jl}\, \vec{x}_{1l}$ $\quad \vec{x}_{1l} \in \beta_1$

$\displaystyle\vec{z}_{2k} = \sum_m C_{km}\, \vec{x}_{2m}$ $\quad \vec{x}_{2m} \in \beta_2$

Then (e_{lj}) is the inverse of (b_{jl}) and $\displaystyle\sum_j e_{lj}\, b_{jl'} = \delta_{ll'}$

Similarly, $\sum_{k} f_{mk} c_{km'} = \delta_{mm'}$.

$$\therefore \quad 0 = \sum_{l,m} d_{lm} \, \mu(\vec{x}_{1l}, \vec{x}_{2m})$$

$$= \sum_{l,m} d_{lm} \, \mu\left(\sum_{j} e_{lj} \vec{z}_{1j}, \sum_{k} f_{mk} \vec{z}_{2k} \right)$$

$$= \sum_{l,m} \sum_{j,k} d_{lm} \, e_{lj} \, f_{mk} \, \mu(\vec{z}_{1j}, \vec{z}_{2k})$$

$\because$ $\mu(\gamma_1 \times \gamma_2)$ is L.I.

$$\therefore \sum_{l,m} d_{lm} \, e_{lj} \, f_{mk} = 0 \quad \text{for } \forall j, k$$

Now, $d_{\ell' m'} = \sum_{\ell, m} d_{\ell m} \delta_{\ell \ell'} \delta_{m m'}$

$$= \sum_{\ell, m} d_{\ell m} \left(\sum_j b_{j \ell'} e_{\ell j} \right) \left(\sum_k c_{k m'} f_{m k} \right)$$

$$= \sum_{j, k} b_{j \ell'} c_{k m'} \underbrace{\left(\sum_{\ell, m} d_{\ell m} e_{\ell j} f_{m k} \right)}_{0}$$

$$= 0$$

for all ℓ', m'.

$\therefore$ $M(\beta_1 \times \beta_2)$ is L. I.

<u>Example</u>: • Let $V = P(F)$. Then: $V \otimes V = P(F^2)$ = space of polynomials in two variables under the product defined to be $f(x) \otimes g(x) = f(x_1) \cdot g(x_2)$.

(∵ let $\beta = \{1, x, x^2, \ldots\}$ be a basis for V.

Then: $\beta \otimes \beta = \{ x^i \otimes x^j : i, j = 0, 1, 2, \ldots \}$

$= \{ x_1^i x_2^j : i, j = 0, 1, 2, \ldots \}$

is a basis for $P(F^2)$.)

- If V is any vector space over F, then under $\otimes$ as scalar multiplication ($\vec{v} \otimes a = a\vec{v}$),

$$V \otimes F = V$$