Lecture 11:

Change of coordinates

Prop: Let β and β' be two ordered bases for a finite-dim. vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$.

$$V_{\beta'} \xrightarrow{I_V} V_{\beta}$$

Then: (a) Q is invertible

(b) For all $\vec{v} \in V$, $[\vec{v}]_\beta = Q[\vec{v}]_{\beta'}$

Proof: (a) Since I_V is invertible, Q is invertible.

(b) Let $\vec{v} \in V$. Then: $[\vec{v}]_\beta = [I_V(\vec{v})]_\beta = \underbrace{[I_V]_{\beta'}^{\beta}}_{=Q} [\vec{v}]_{\beta'}$

Def: The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called the change of coordinate matrix from β' to β.

<u>Remark:</u> To compute $Q = [I_V]_{\beta'}^{\beta}$,

if $\beta = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ and $\beta' = \{\vec{x}_1{}', \vec{x}_2{}', \ldots, \vec{x}_n{}'\}$,

then:

$$Q = \begin{pmatrix} | & & \\ [I_V(\vec{x}_1{}')]_\beta & \cdots & \cdots \\ | & & \end{pmatrix}$$

$$= \begin{pmatrix} | & & | & \\ [\vec{x}_1{}']_\beta & \cdots & [\vec{x}_j{}']_\beta & \cdots \\ | & & | & \end{pmatrix}$$

Proposition: Let T be a linear operator on finite-dim V. Let β and β' be ordered bases of V. Suppose $Q = [I_V]_{\beta'}^{\beta}$.

Then:
$$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

$$\underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V} \rightsquigarrow [T]_\beta$$

$$\underset{\beta'}{V} \xrightarrow{T} \underset{\beta'}{V} \rightsquigarrow [T]_{\beta'}$$

Proof:
$$\begin{aligned} Q[T]_{\beta'} &= [I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [I_V \circ T]_{\beta'}^{\beta} \\ &= [T \circ I_V]_{\beta'}^{\beta} \\ &= [T]_{\beta}^{\beta}[I_V]_{\beta'}^{\beta} \\ &= [T]_\beta Q \end{aligned}$$

$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V}$$

Remark: A linear $T: V \to V$ is called linear operator.

Corollary: Let $A \in M_{n \times n}(F)$ and let $\gamma = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ be an ordered basis for F^n.

Then: $[L_A]_\gamma = Q^{-1} A Q$, $Q = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix}$

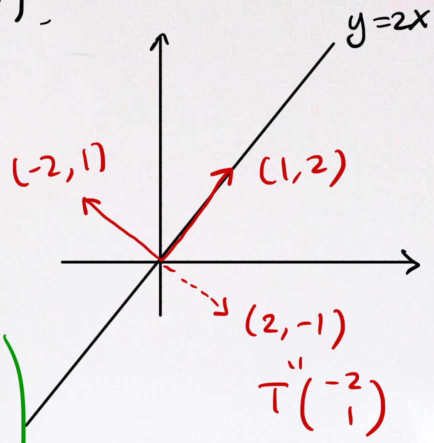
$\Leftrightarrow [L_A]_\gamma = Q^{-1} [L_A]_\beta Q$

standard ordered basis.

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = 2x$.

Want to compute $[T]_\beta$, where $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Consider $\beta' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$



- $$[T]_{\beta'} = \left(\left[T\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]_{\beta'} \;\; \left[T\begin{pmatrix} -2 \\ 1 \end{pmatrix} \right]_{\beta'} \right) = \left(\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]_{\beta'} \;\; \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} \right]_{\beta'} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $$Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \Rightarrow Q^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\therefore \quad [T]_{\beta'} = Q^{-1}[T]_{\beta}Q$$

$$\Longleftrightarrow \quad [T]_{\beta} = Q \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\shortparallel}{[T]_{\beta'}}} Q^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

<u>Def:</u> Given two matrices $A, B \in M_{n\times n}(F)$.
We say B is <u>similar</u> to A if $\exists\, Q \in M_{n\times n}$ s.t.

$$B = Q^{-1} A Q.$$

Lecture 12:

Def: A linear operator $T: V \to V$ (where V is finite-dim) is called <u>diagonalizable</u> if $\exists$ an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.
A square matrix A is called diagonalizable if L_A is so.

Observation: Say $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

If $\underset{(D_{ij})}{D} = [T]_\beta$ is diagonal, then $\forall \vec{v}_j \in \beta$, we have:

$$T(\vec{v}_j) = \sum_{i=1}^{n} D_{ij} \vec{v}_i = \underbrace{D_{jj}}_{\lambda_j} \vec{v}_j = \lambda_j \vec{v}_j$$

Conversely, if $T(\vec{v}_j) = \lambda_j \vec{v}_j$ for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$,

then: $[T]_\beta = \begin{pmatrix} | & & \\ [T(\vec{v}_1)]_\beta & - & - \\ | & & \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & \lambda_n \end{pmatrix}$

Def: Let T be a linear operator on a vector space V/F.
A non-zero vector $\vec{v} \in V$ is called an eigenvector of T
if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \vec{v}$. In this case, $\lambda \in F$
is called an eigenvalue corresponding to the eigenvector $\vec{v}$.

For a square matrix $A \in M_{n\times n}(F)$, a non-zero vector $\vec{v} \in F^n$
is called an eigenvector of A if it is an eigenvector of L_A.

That is: $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in F$.

λ is called the eigenvalue corresponding to the eigenvector $\vec{v}$.

Prop: A linear operator $T: V \to V$ (V = fin-dim) is diagonalizable iff $\exists$ an ordered basis β for V consisting of eigenvectors of T.

In such case, if $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then:

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

where λ_j is the eigenvalue of T corresponding to $\vec{v}_j$.

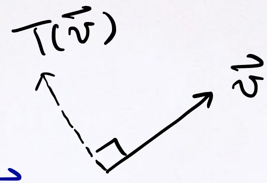
<u>Example:</u> • $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\beta = \underbrace{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}}$

Check that they are All eigenvectors and β is basis.

Then: $[L_A]_\beta = \begin{pmatrix} 4 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

• Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\frac{\pi}{2}$ in counter-clockwise direction.

(Check: $T = L_A$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)



Then: $T(\vec{v})$ is always perpendicular to $\vec{v}$.

$\therefore$ For $\forall \vec{v} \neq \vec{0}$, it cannot be an eigenvector because:

$T(\vec{v}) \neq \lambda \vec{v}$ for some $\lambda \in F$

<u>Example</u>: Consider $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by:

Space of smooth function

are infinitely differentiable

$$T(f) = f'$$

Then an eigenvector of T with eigenvalue λ is a non-zero solution of:

$$\frac{df}{dt} = \lambda f(t)$$

$$\iff f(t) = C e^{\lambda t} \text{ for some constant } C.$$

$\therefore$ all $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Def: The characteristic polynomial of $A \in M_{n\times n}(F)$ is defined as the polynomial $f_A(t) \overset{def}{=} \det(A - t I_n) \in P_n(F)$

Def: Let T be a linear operator on an n-dim vector space V. Choose an ordered basis β for V. Then, the characteristic polynomial of T is defined as the characteristic polynomial of $[T]_\beta$.

(i.e. $f_T(t) \overset{def}{=} \det([T]_\beta - t I_n) \in P_n(F)$)