# Dual Space

Let $V$ be a vector space over $F$.

**Definition:** A linear functional on $V$ is a linear map $f: V \to F$.

**Remark:** A linear functional belongs to $\mathcal{L}(V, F)$.

**Definition:** The dual space, denoted by $V^*$, is the space of all linear functional on $V$. That is, $V^* = \mathcal{L}(V, F)$.

**Proposition:** Suppose $V$ is finite-dimensional. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of $V$. For each $i = 1, 2, \ldots, n$, define a linear functional $f_i$ by setting: $f_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Then: $\{f_1, f_2, \ldots, f_n\}$ is a basis of $V^*$, called the dual basis of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. $\therefore \dim(V) = \dim(V^*)$

<u>Proof:</u> • $\{f_1, f_2, \dots, f_n\}$ is linearly independent.

Suppose: $a_1 f_1 + a_2 f_2 + \dots + a_n f_n = 0 \leftarrow$ zero functional

For each $\vec{v}_i$,

$$(a_1 f_1 + \dots + a_n f_n)(\vec{v}_i) = 0 \Rightarrow a_1 f_1(\vec{v}_1) + \dots + a_n f_n(\vec{v}_i) = 0$$

$$\Rightarrow a_i = 0.$$

$\therefore \{f_1, f_2, \dots, f_n\}$ is linearly independent.

• $\operatorname{Span}(\{f_1, f_2, \dots, f_n\}) = V^*$.

Let $f \in V^*$. Suppose $f(\vec{v}_i) = b_i$.

Claim: $b_1 f_1 + b_2 f_2 + \dots + b_n f_n = f$.

Check: For each $\vec{v}_i$,

$$(b_1 f_1 + \dots + b_n f_n)(\vec{v}_i) = b_i \cancelto{1}{f_i(\vec{v}_i)} = f(\vec{v}_i) \Rightarrow b_1 f_1 + \dots + b_n f_n = f.$$

Example: Let $\beta = \{1+x, 1-x, x^2\}$ be the ordered basis for $P_2(\mathbb{R})$

Let $\beta^*$ be the dual basis of $\beta$.
$\quad\;\; = \{f_1, f_2, f_3\}$

Then:

$$1 = f_1(1+x) = f_1(1) + f_1(x)$$

$$0 = f_1(1-x) = f_1(1) - f_1(x)$$

$$0 = f_1(x^2)$$

Solving: we get $f_1(1) = \frac{1}{2}$, $f_1(x) = \frac{1}{2}$, $f_1(x^2) = 0$

Thus, $f_1(a + bx + cx^2) = a f_1(1) + b f_1(x) + c f_1(x^2)$

$$= \frac{1}{2}a + \frac{1}{2}b$$

$f_2$ and $f_3$ can be computed similarly

**Remark:**

- $\dim(V) = \dim(V^*)$ $\therefore$ $V$ is isomorphic to $V^*$

  fin-dim

- $V^{**} = (V^*)^* =$ dual of the dual space

**Proposition:** Suppose $V$ is fin-dim. The map $\ell : V \to V^{**}$ defined by $\ell(\vec{v})(f) \overset{\text{def}}{=} f(\vec{v})$ is an isomorphism.

$\in V^*$

**Proof:** $\ell$ is linear (Exercise)

To prove that $\ell$ is an isomorphism, we can just show that $\vec{v}^*$ is 1-1 (since $\dim(V) = \dim(V^{**})$)

Suppose $\ell(\vec{v}) = 0$ in $V^{**}$.

$\Rightarrow \ell(\vec{v})(f) = 0$ for all $f \in V^*$

Then: $f(\vec{v}) = 0$ for all $f \in V^*$

The only possibility is $\vec{v} = \vec{0}$.

$\therefore \mathrm{Null}(\ell) = \{\vec{0}\}$.

Thus, $\ell$ is 1-1 and onto.

(isomorphism)

**Definition:** Let $T: V \to W$ be linear. The dual map (or transpose) of $T$ is the map $T^*: W^* \to V^*$ defined by:

$$T^*(g) = g(T) \quad \text{for all } g \in W^*.$$

**Proposition:** Suppose $V$ is fin-dimensional. Let $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of $V$. Let $\beta^* = \{f_1, \dots, f_n\}$ be the dual basis of $\beta$. Let $T: V \to W$ and $\gamma$ be the basis of $W$. Denote the dual basis of $\gamma$ by $\gamma^*$. Then: (1) $T^*$ is linear

(2) $$\underline{[T^*]_{\gamma^*}^{\beta^*}} = \underbrace{\left([T]_{\beta}^{\gamma}\right)^T}_{\text{Matrix transpose}}$$

Transpose of $T$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \beta & & \gamma \\ & & \\ V^* & \xleftarrow{T^*} & W^* \\ \beta^* & & \gamma^* \end{array}$$

Proof: For any $g \in W^*$, $T^*(g) = \underbrace{g}_{\text{linear}} \circ \underbrace{T}_{\text{Linear}}$ is linear.

$\therefore$ $T^*(g)$ is a linear functional on $V$. $\therefore$ $T^*(g) \in V^*$.

Thus: $T^*$ maps $W^*$ to $V^*$.

$T^*$ is linear: $T^*(\alpha g_1 + g_2) = (\alpha g_1 + g_2) \circ T$

$= \alpha g_1 \circ T + g_2 \circ T = \alpha T^*(g_1) + T^*(g_2)$

Now, write $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$

$\beta^* = \{f_1, f_2, \ldots, f_n\}$

$\gamma^* = \{g_1, g_2, \ldots, g_n\}$

Let $A = [T]_\beta^\gamma = (a_{ij})$

To find the $j^{th}$ col of $[T^*]_{\gamma^*}^{\beta^*}$, we write:

$T^*(g_j)$ as a lin. combination of $f_1, f_2, \ldots, f_n$.

Now, $T^*(g_j) = g_j \circ T = \sum_{i=1} (g_j \circ T)(\vec{v}_i) f_i$

$\therefore$ the $i^{th}$-row, $j^{th}$ col entry of $[T^*]_{\gamma^*}^{\beta^*}$ is given by:

$$g_j \circ T(\vec{v}_i) = g_j\left(\sum_{k=1}^{m} A_{ki} \vec{w}_k\right)$$

$$= \sum_{k=1}^{m} A_{ki}\, g_j(\vec{w}_k) = A_{ji}$$

$\therefore [T^*]_{\gamma^*}^{\beta^*} = A^T = [T]_\beta^\gamma$.