

Dual Space Let  $V$  be a vector space over  $F$ .

Definition: A linear functional on  $V$  is a linear map  $f: V \rightarrow F$ .

Remark: A linear functional belongs to  $\mathcal{L}(V, F)$ .

Definition: The dual space, denoted by  $V^*$ , is the space of all linear functional on  $V$ . That is,  $V^* = \mathcal{L}(V, F)$ .

Proposition: Suppose  $V$  is finite-dimensional. Let  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  be a basis of  $V$ . For each  $i = 1, 2, \dots, n$ , define a linear functional  $f_i$  by setting:  $f_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ .

Then:  $\{f_1, f_2, \dots, f_n\}$  is a basis of  $V^*$ , called the dual basis of  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ .  $\therefore \dim(V) = \dim(V^*)$

Proof: •  $\{f_1, f_2, \dots, f_n\}$  is linearly independent.

Suppose:  $a_1 f_1 + a_2 f_2 + \dots + a_n f_n = 0$  ← zero functional

For each  $\vec{v}_i$ ,

$$(a_1 f_1 + \dots + a_n f_n)(\vec{v}_i) = 0 \Rightarrow a_1 f_1(\vec{v}_i) + \dots + a_n f_n(\vec{v}_i) = 0$$
$$\Rightarrow a_i = 0.$$

∴  $\{f_1, f_2, \dots, f_n\}$  is linearly independent.

•  $\text{Span}(\{f_1, f_2, \dots, f_n\}) = V^*$ .

Let  $f \in V^*$ . Suppose  $f(\vec{v}_i) = b_i$ .

Claim:  $b_1 f_1 + b_2 f_2 + \dots + b_n f_n = f$ .

Check: For each  $\vec{v}_i$ ,

$$(b_1 f_1 + \dots + b_n f_n)(\vec{v}_i) = b_i f_i(\vec{v}_i) = f(\vec{v}_i) \Rightarrow b_1 f_1 + \dots + b_n f_n = f.$$

Example: Let  $\beta = \{1+x, 1-x, x^2\}$  be the ordered basis for  $P_2(\mathbb{R})$

Let  $\beta^*$  be the dual basis of  $\beta$ .

$\{f_1, f_2, f_3\}$

$$\text{Then: } 1 = f_1(1+x) = f_1(1) + f_1(x)$$

$$0 = f_1(1-x) = f_1(1) - f_1(x)$$

$$0 = f_1(x^2)$$

Solving: we get  $f_1(1) = \frac{1}{2}$ ,  $f_1(x) = \frac{1}{2}$ ,  $f_1(x^2) = 0$

$$\begin{aligned} \text{Thus, } f_1(a+bx+cx^2) &= a f_1(1) + b f_1(x) + c f_1(x^2) \\ &= \frac{1}{2}a + \frac{1}{2}b \end{aligned}$$

$f_2$  and  $f_3$  can be computed similarly.

- Remark:
- $\dim(V) = \dim(V^*) \quad \therefore V$  is isomorphic to  $V^*$   
    ↑  
    fin-dim
  - $V^{**} = (V^*)^* =$  dual of the dual space

Proposition: Suppose  $V$  is fin-dim. The map  $\ell: V \rightarrow V^{**}$  defined by  $\ell(\vec{v})(f) \stackrel{\text{def}}{=} f(\vec{v})$  is an isomorphism.

Proof:  $\ell$  is linear (Exercise)

To prove that  $\ell$  is an isomorphism, we can just show that  $\vec{v}^*$  is 1-1 (since  $\dim(V) = \dim(V^{**})$ )

Suppose  $l(\vec{v}) = 0$  in  $V^{**}$ .

$\Rightarrow l(\vec{v})(f) = 0$  for all  $f \in V^*$

Then:  $f(\vec{v}) = 0$  for all  $f \in V^*$

The only possibility is  $\vec{v} = \vec{0}$ .

$\therefore \text{Null}(l) = \{\vec{0}\}$ .

Thus,  $l$  is 1-1 and onto.  
(isomorphism)

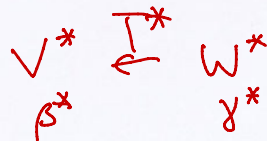
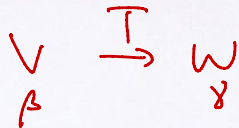
Definition: Let  $T: V \rightarrow W$  be linear. The dual map (or transpose) of  $T$  is the map  $T^*: W^* \rightarrow V^*$  defined by:

$$T^*(g) = g(T) \text{ for all } g \in W^*.$$

Proposition: Suppose  $V$  is fin-dimensional. Let  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  be a basis of  $V$ . Let  $\beta^* = \{f_1, \dots, f_n\}$  be the dual basis of  $\beta$ .

Let  $T: V \rightarrow W$  and  $\gamma$  be the basis of  $W$ . Denote the dual basis of  $\gamma$  by  $\gamma^*$ . Then: (1)  $T^*$  is linear

$$(2) \quad \underbrace{[T^*]_{\beta^*}^{\gamma^*}}_{\text{Transpose of } T} = \underbrace{([T]_{\gamma}^{\beta})^T}_{\text{Matrix transpose}}$$



Proof: For any  $g \in W^*$ ,  $T^*(g) = \underbrace{g \circ T}$  is linear.

$\therefore T^*(g)$  is a linear functional on  $V$ .  $\therefore T^*(g) \in V^*$ .

Thus:  $T^*$  maps  $W^*$  to  $V^*$ .

$$\begin{aligned} T^* \text{ is linear: } T^*(\alpha g_1 + g_2) &= (\alpha g_1 + g_2) \circ T \\ &= \alpha g_1 \circ T + g_2 \circ T = \alpha T^*(g_1) + T^*(g_2) \end{aligned}$$

Now, write  $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$

$\beta^* = \{f_1, f_2, \dots, f_n\}$

$\gamma^* = \{g_1, g_2, \dots, g_n\}$

$$\text{Let } A = [T]_{\beta}^{\gamma} = (a_{ij})$$

To find the  $j^{\text{th}}$  col of  $[T^*]_{\gamma^*}^{\beta^*}$ , we write:

$T^*(g_j)$  as a lin. combination of  $f_1, f_2, \dots, f_n$ .

$$\text{Now, } T^*(g_j) = g_j \circ T = \sum_{i=1}^m (g_j \circ T)(\vec{v}_i) f_i$$

$\therefore$  the  $i^{\text{th}}$ -row,  $j^{\text{th}}$  col entry of  $[T^*]_{\gamma^*}^{\beta^*}$  is given by:

$$\begin{aligned} g_j \circ T(\vec{v}_i) &= g_j \left( \sum_{k=1}^m A_{ki} \vec{w}_k \right) \\ &= \sum_{k=1}^m A_{ki} g_j(\vec{w}_k) = A_{ji} \end{aligned}$$

$$\therefore [T^*]_{\gamma^*}^{\beta^*} = A^T = [T]_{\beta}^{\gamma}$$