Recall

Basic concepts

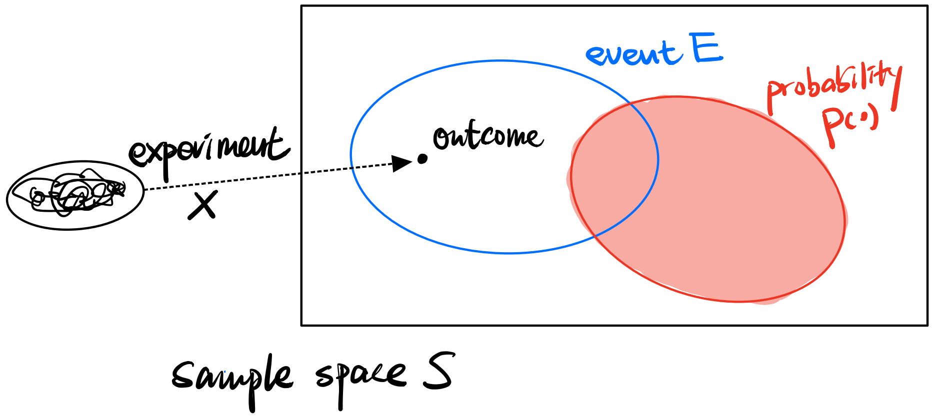

Figure 1: A logic diagram of basic concepts

Axioms of probability (Kolmogorov)

Axiom 1: $0 \le P(E) \le 1$; **Axiom 2:** $P(S) = 1$; **Axiom 3:** For **disjoint** (mutually exclusive) events $(E_n)_{n=1}^{\infty}$, $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$.

From the above axioms, we can deduce the following properties of probability $P(\cdot)$. Moreover, later we shall justify that the relative frequency 'definition' of probability is almost true.

Basic properties of $P(\cdot)$

- $P(\emptyset) = 0$
- $P(E^c) = 1 - P(E)$
- (monotone) $P(E) \le P(F)$ if $E \subset F$.
- (inclusion-exclusion) $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i < j \le n} P(E_i E_j) + \cdots + (-1)^{n+1} P(E_1 \cdots E_n)$
- (finite additive) For disjoint $(E_i)_{i=1}^{n}$, $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$
- (countable subadditive) $P(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} P(E_n)$
- (continuous) Let $(E_n)_{n=1}^{\infty}$ be a sequence of events.

$$\begin{cases} E_n \subset E_{n+1} \implies P(\lim_{n\to\infty} E_n) = P(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n) \\ E_n \supset E_{n+1} \implies P(\lim_{n\to\infty} E_n) = P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n) \end{cases}$$

Basic concepts

Example 1. Roll a die repeatedly until the first 6 appears and then we stop the experiment.

(a) What's the sample space?

(b) Let $n \geq 1$. Explicitly describe the event E_n that we roll the die for $\leq n$ times and stop.

(c) What's the $(\bigcup_{n=1}^{\infty} E_n)^c$?

Solution. (a) The sample space

$$S = \bigcup_{k=1}^{\infty} \Big\{(i_1, \dots, i_k, 6) \colon i_1, \dots i_k \in \{1, \dots 5\}\Big\} \bigcup \Big\{(i_k)_{k=1}^{\infty} \colon \forall\, k \in \mathbb{N}, i_k \in \{1, \dots 5\}\Big\}$$

with convention $\{(i_1, i_0, 6)\} = \{(6)\}$. The first part represents the outcomes that we stop the experiment after rolling finite times while the last set consists of the outcomes that we never stop.

(b) Similarly, $E_n = \bigcup_{k=1}^{n-1} \Big\{(i_1, \dots, i_k, 6) \colon i_1, \dots i_k \in \{1, \dots 5\}\Big\}$.

(c) From the expressions of S and E_n, we have

$$(\bigcup_{n=1}^{\infty} E_n)^c = \Big\{(i_k)_{k=1}^{\infty} \colon \forall\, k \in \mathbb{N}, i_k \in \{1, \dots 5\}\Big\}$$

which is the event that we never stop the experiment. □

Until now, except the sample spaces of outcomes with equal probabilities, we do not know too many concrete examples of probabilities (satisfying the axioms). What is the natural probability on the sample space of Example 1?

Sample spaces with equally likely outcomes

Example 2. Roll a die twice. What's the probability that the second number is larger than the first?

Solution. Explicitly write down the sample space $S = \Big\{(i, j) \colon i, j \in \{1, \dots, 6\}\Big\}$ and the event $E = \Big\{(i, j) \colon i < j\Big\} = \{(1, 2), \dots (1, 5), \dots, (5, 6)\}$. Then $|S| = 36$ and $|E| = 5 + 4 + \cdots + 1 = 15$. By the assumption on equal probabilities, we have $P(E) = 15/36 = 5/12$. □

Example 3. In a game, the total 52 (4×13) cards are dealt out to 4 players. What's the probability of

(a) the event A that one of the players receives all 13 heart ♡ cards?

(b) the event B that each player receives 1 aces?

(c) the event C that each player receives at least 1 heart ♡ cards?

Solution. (a) Let $E_i, (i = 1, \dots, 4)$ be the event that the player i receives 13 hearts. Then $P(E_i) = 1/\binom{52}{13}$ which can the obtained by reasoning: choose 13 cards for the player i from 52 cards, only 1 selection consists of all heart cards.

Since there are only 13 heart cards, any two players can not have all heart cards at the same time, i.e., E_i are disjoint. By finite additivity,

$$P(A) = P(\bigcup_{i=1}^{4} E_i) = \sum_{i=1}^{4} P(E_i) = \frac{4}{\binom{52}{13}} \approx 6.3 \times 10^{-12}.$$

(b) There are $\binom{52}{13,13,13,13}$ ways of dealing out 52 cards to 4 players with equal probabilities. To determine the outcomes making event B happen, we first determine the positions of 4 aces which results in 4! permutations, then we counting the ways to distribute the remaining $52 - 4 = 48$ cards to the 4 players. Hence

$$P(B) = \frac{4! \times \binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} \approx 0.1055.$$

(c) Let $C_i, (i = 1, \dots, 4)$ denote the event that the player i does not receive heart cards. By complement,

$$P(C) = 1 - P(\bigcup_{i=1}^{4} C_i).$$

To obtain $P(\bigcup_{i=1}^{4} C_i)$, we will use the inclusion-exclusion principle. It follows from the similar arguments of (a) that

$$i = 1, \dots 4 \qquad P(C_i) = \frac{\binom{39}{13}}{\binom{52}{13}}$$

$$1 \le i < j \le 4 \qquad P(C_iC_j) = \frac{\binom{39}{26}}{\binom{52}{26}}$$

$$1 \le i < j < k \le 4 \qquad P(C_iC_jC_k) = \frac{\binom{39}{39}}{\binom{52}{39}}.$$

Notice $P(C_1C_2C_3C_4) = 0$ since the 4 players cannot avoid having heard cards at the same time. Hence by inclusion-exclusion,

$$P(\bigcup_{i=1}^{4} C_i) = 4 \times \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \times \frac{\binom{39}{26}}{\binom{52}{26}} + \binom{4}{3} \times \frac{\binom{39}{39}}{\binom{52}{39}} - 0.$$

Thus $P(C) = 1 - P(\bigcup_{i=1}^{4} C_i) \approx 0.9488.$

□