THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2022-23 Term 1
Solutions to Midterm Examination

1 (10 pts)

There is a 65% chance that event A will occur. If A does not occur, then there is a 10% chance that B will occur. What is the probability that at least one of the events A or B will occur?

***Solution*.** By assumption,

$$P(A) = 0.65 \quad \text{and} \quad P(B \mid A^c) = 0.1.$$

Then

$$P(A^c) = 1 - P(A) = 0.35$$

and

$$P(B \cap A^c) = P(B \mid A^c)P(A^c) = 0.1 \times 0.35 = 0.035.$$

Hence the probability that at least one of the events A or B will occur is

$$\begin{aligned} P(A \cup B) = P\left(A \cup (B \cap A^c)\right) = P(A) + P(B \cap A^c) = 0.65 + 0.035 &= 0.685 \\ &= 68.5\%. \end{aligned}$$

□

2 (15 pts)

Five cards are randomly chosen, without replacement, from an ordinary deck of 52 playing cards.

(a) (5 pts) Compute the probability that the chosen 5 cards contain exactly 2 aces.

(b) (5 pts) Compute the probability that the chosen cards have 5 different ranks.

(c) (5 pts) Compute the probability that the chosen 5 cards contain three cards of one rank and two cards of another rank.

Solution.

(a) Let S be the event that contains all the outcomes of the 5 chosen cards, E be the event that the chosen 5 cards contain exactly 2 aces. Since we are going to choose 5 cards from the total 52 cards. Then $|S| = \binom{52}{5}$. When the event E happens, we need to choose 2 aces from the total 4 aces, and choose the other 3 cards from the remaining 48 cards. Then $|E| = \binom{4}{2}\binom{48}{3}$. It follows that

$$P(E) = \frac{|E|}{|S|} = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}.$$

(b) Let F be the event that the chosen cards have 5 different ranks. When the event F happens, first, we need to choose 5 ranks from the total 13 ranks. Then since each rank has 4 cards, we have 4^5 possible choices to choose the cards if we have selected the ranks. It means that $|F| = \binom{13}{5} \times 4^5$. Therefore,

$$P(F) = \frac{|F|}{|S|} = \frac{\binom{13}{5} \times 4^5}{\binom{52}{5}}.$$

(c) Firstly we count the **ordered** pairs of ranks from the total 13 ranks. There are 13×12 such ordered pairs. Secondly, for each ordered pair of ranks, we choose 3 cards from 4 cards of the first rank and 2 cards from 4 cards of the second rank. There are $\binom{4}{3}\binom{4}{2}$ such choices. Hence target the probability is

$$\frac{(13 \times 12) \times (\binom{4}{3}\binom{4}{2})}{\binom{52}{5}}.$$

□

3 (12 pts)

A coin having probability p of landing on heads is tossed repeatedly until it comes up to the fifth head. Let X denote the numbers of times we have to toss the coin until it comes up the fifth head.

(a) (6 pts) Calculate $P\{X = 5\}$ and $P\{X = 6\}$.

(b) (6 pts) Calculate $P\{X = n\}$ for integers $n \geq 5$.

Solution.

(a) The event $\{X = 5\}$ means that the 5-th trial shows head and all the 4 previous trials show heads. Then

$$P\{X = 5\} = p \cdot p^4 = p^5.$$

The event $\{X = 6\}$ means that the 6-th trial shows head and 4 out of the 5 previous trials show heads. Then

$$P\{X = 6\} = p \cdot \binom{5}{4} p^4(1-p) = 5p^5(1-p).$$

(b) The event $\{X = n\}$ means that the n-th trial shows head and 4 out of the $(n-1)$ previous trials show heads. Then

$$P\{X = n\} = p \cdot \binom{n-1}{4} p^4(1-p)^{n-1-4} = \binom{n-1}{4} p^5(1-p)^{n-5}.$$

□

4 (14 pts)

Let X be a binomial random variable with parameters n and p.

(a) (9 pts) Find $E[X^k]$, $k = 1, 2, 3$.

(b) (5 pts) Find $\mathrm{Var}(3X + 2)$.

Solution.

(a) Let Y be a binomial random variable with parameters $n - 1$ and p. If $n = 1$, by convention we set Y to be 0 with probability 1. Then for $k \in \mathbb{N}$,

$$\begin{aligned}
E[X^k] &= \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} np^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{(n-1)-(i-1)} \\
&= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \qquad \text{by changing index } j = i - 1 \\
&= np\, E[(Y+1)^{k-1}].
\end{aligned}$$

- Take $k = 1$. Then $E[X] = np\, E[1] = np$.
- Take $k = 2$. Then

$$E[X^2] = np\, E[Y + 1] = np\,((n-1)p + 1) = n^2p^2 - np^2 + np.$$

- Note that $E[Y^2] = (n-1)p((n-2)p+1)$ and $E[Y] = (n-1)p$ by previous arguments. Take $k = 3$. Then

$$\begin{aligned}
E[X^3] &= np \cdot E[(Y+1)^2] = np(E[Y^2] + 2E[Y] + 1) \\
&= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np.
\end{aligned}$$

(b) By the linearity of expectation,

$$\begin{aligned}
\mathrm{Var}(3X+2) &= E[(3X+2)^2] - (E[3X+2])^2 \\
&= E[9X^2 + 12X + 4] - (3E[X] + 2)^2 \\
&= 9E[X^2] + 12E[X] + 4 - 9(E[X])^2 - 12E[X] - 4 \\
&= 9(E[X^2] - (E[X])^2) \\
&= 9(n^2p^2 - np^2 + np - n^2p^2) \\
&= 9np(1-p).
\end{aligned}$$

□

5 (15 pts)

Let X be a continuous random variable, having a density function given by

$$f(x) = \begin{cases} c(1-x^2), & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) (5 pts) What is the value of c?

(ii) (5 pts) What is the cumulative distribution function F of X? where $F(a) = P\{X \le a\}$.

(iii) (5 pts) Find $P\{X > 1/2\}$.

Solution.

(i) Since

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} c(1-x^2)\,dx = \frac{4}{3}c,$$

we have

$$c = \frac{3}{4}.$$

(ii) For $a \in \mathbb{R}$,

$$F(a) = P\{X \le a\} = \int_{-\infty}^{a} f(x)\,dx.$$

Then for $a \in (-1, 1)$,

$$F(a) = \int_{-1}^{a} \frac{3}{4}(1-x^2)\,dx = -\frac{a^3}{4} + \frac{3a}{4} + \frac{1}{2} = \frac{-a^3+3a+2}{4}.$$

Hence

$$F(a) = \begin{cases} 0 & \text{if } a \le -1 \\ \frac{-a^3+3a+2}{4} & \text{if } a \in (-1,1) \\ 1 & \text{if } a \ge 1. \end{cases}$$

(iii) By (ii),

$$P\{X > 1/2\} = 1 - P\{X \le 1/2\} = 1 - F(1/2) = 1 - \frac{27}{32} = \frac{5}{32} = 0.15625.$$

□

6 (10 pts)

Six balls are to be randomly chosen without replacement from an urn containing 8 red, 10 green, and 12 blue balls.

(a) (5 pts) What is the probability at least two red balls are chosen?

(b) (5 pts) Given that no red balls are chosen, what is the conditional probability that there are exactly 3 green balls among the 6 chosen?

Solution.

(a) Let E be the event that at lest two red balls are chosen, F_1 be the event that only one red ball is chosen, F_2 be the event that no red balls are chosen. Note that $E = (F_1 \cup F_2)^c$ and $F_1 \cap F_2 = \varnothing$. Let S be the whole sample space containing all the outcomes of the chosen 6 balls. Since we are going to choose 6 balls among total 30 balls, then $|S| = \binom{30}{6}$.

If the 6 chosen balls contain only one red ball, then we need to chose one red ball among total 8 red balls, and choose another 5 balls among the remaining 22 balls. It means that $|F_1| = \binom{8}{1}\binom{22}{5}$.

If the 6 chosen balls contain no red balls, then we need to choose them from the 22 balls which are not red. It means that $|F_2| = \binom{22}{6}$.

Therefore,

$$P(E) = 1 - P(F_1 \cup F_2) = 1 - P(F_1) - P(F_2) = 1 - \frac{|F_1|}{|S|} - \frac{|F_2|}{|S|} = 1 - \frac{\binom{8}{1}\binom{22}{5}}{\binom{30}{6}} - \frac{\binom{22}{6}}{\binom{30}{6}}.$$

(b) We still let S be the whole sample space containing all the outcomes of the chosen 6 balls. And let B be the event that no red balls are chosen, C be the event that there are exactly 3 green balls. We can see that $B \cap C$ is the event that the chosen 6 balls contain exactly 3 green balls and 3 blue balls. When the event $B \cap C$ happens, we need to choose 3 green balls from the total 10 green balls and choose 3 blue balls from the total 12 blue balls. It means that $|B \cap C| = \binom{10}{3}\binom{12}{3}$. From the previous part, we have known $|B| = \binom{22}{6}$.

In conclusion,

$$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{\frac{|B \cap C|}{|S|}}{\frac{|B|}{|S|}} = \frac{|B \cap C|}{|B|} = \frac{\binom{10}{3}\binom{12}{3}}{\binom{22}{6}}.$$

□

7 (14 pts)

Let Z be a standard normal random variable.

(a) (7 pts) Find the probability density function of $X = Z^2 + 1$;

(b) (7 pts) Find $E[Y]$ for $Y = (Z + 1)^2$.

Solution. Let F denote the cumulative distribution function of X and f denote the probability density function of X. Let Φ be the cumulative distribution function of Z.

(a) For $t > 1$, since $\Phi(x) = 1 - \Phi(-x)$ for $x \in \mathbb{R}$,

$$\begin{aligned} F(t) = P\{X \le t\} = P\{Z^2 + 1 \le t\} &= P\{-\sqrt{t-1} \le Z \le \sqrt{t-1}\} \\ &= \Phi(\sqrt{t-1}) - \Phi(-\sqrt{t-1}) \\ &= 2\Phi(\sqrt{t-1}) - 1. \end{aligned}$$

Then by differentiation and the chain rule,

$$f(t) = F'(t) = 2\Phi'(\sqrt{t-1}) \cdot \frac{1}{2\sqrt{t-1}} = \frac{1}{\sqrt{2\pi(t-1)}} e^{-(t-1)/2}.$$

For $t < 1$, since $Z^2 + 1 \ge 1$,

$$F(t) = P\{Z^2 + 1 \le t\} = 0,$$

and so $f(t) = F'(t) = 0$. Hence there is a probability density function

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi(x-1)}} e^{-(x-1)/2} & \text{if } x > 1 \\ 0 & \text{if } x \le 1. \end{cases}$$

(b) Note that $E[Z] = 0$ and $E[Z^2] = \text{Var}(Z) + E[Z]^2 = 1 + 0 = 1$. By the linearity of expectation,

$$E[Y] = E[(Z+1)^2] = E[Z^2 + 2Z + 1] = E[Z^2] + 2E[Z] + 1 = 1 + 0 + 1 = 2.$$

□

8 (10 pts)

Prove that for any events $E_1, \ldots, E_n$,

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) \geq P(E_1) + \cdots + P(E_n) - (n-1). \tag{1}$$

Solution 1. Note $P(A^c) = 1 - P(A)$ for event A. Then by the subadditivity of probability,

$$P(E_1^c \cup E_2^c \cup \cdots \cup E_n^c) \leq \sum_{i=1}^{n} P(E_i^c) = \sum_{i=1}^{n} (1 - P(E_i)) = n - \sum_{i=1}^{n} P(E_i) \tag{2}$$

Hence by De Morgan's law and (2),

$$\begin{aligned} P(E_1 \cap E_2 \cap \cdots \cap E_n) &= 1 - P\left((E_1 \cap E_2 \cap \cdots \cap E_n)^c\right) \\ &= 1 - P(E_1^c \cup E_2^c \cup \cdots \cup E_n^c) \\ &\geq 1 - (n - \sum_{i=1}^{n} P(E_i)) \qquad \text{by (2)} \\ &= \sum_{i=1}^{n} P(E_i) - (n-1). \end{aligned}$$

□

Solution 2. We prove (1) by induction. Note that (1) holds trivially for $n = 1$. Suppose (1) holds for n, that is

$$P(E_1 \cap \cdots \cap E_n) \geq \sum_{i=1}^{n} P(E_n) - (n-1). \tag{3}$$

Since $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$ for events A, B,

$$\begin{aligned} P(E_1 \cap \cdots \cap E_{n+1}) &\geq P\left(\cap_{i=1}^{n} E_i\right) + P(E_{n+1}) - 1 \\ &\geq \sum_{i=1}^{n} P(E_i) - (n-1) + P(E_{n+1}) - 1 \\ &= \sum_{i=1}^{n+1} P(E_i) - n. \end{aligned}$$

Hence (1) holds for $n+1$. This finishes the proof. □

— *THE END* —