Math 3280A

2022-12-1.

Review

- The central limit Thm

Let $X_1, \cdots, X_n, \cdots,$ be an i.i.d. sequence of r.v.'s, each having finite mean μ and variance σ^2.

Then $\forall\, a \in \mathbb{R}$,

$$P\left\{ \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma} \leq a \right\} \longrightarrow \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}}\, dx,$$

as $n \to \infty$.

Example 1 If 10 fair dice are rolled, find the approximate prob. that the sum obtained is between 30 and 40.

Solution: Let X_i be the value obtained in the i-th roll, $i = 1, 2, \cdots, 10$.

We need to calculate

$$P\{ 29.5 \leq X_1 + \cdots + X_{10} \leq 40.5 \}$$

Notice that $\mu = E[X_i] = \frac{1}{6}(1 + 2 + \cdots + 6)$

$$= 7/2$$

$$E[X_i^2] = \frac{1}{6}\left(1^2+2^2+\cdots+6^2\right)$$
$$= \frac{1}{6}\cdot\frac{6\times 7\times 13}{6}$$

$$\sigma^2 = \operatorname{Var}(X_i) = E[X_i^2] - E[X_i]^2$$
$$= \frac{35}{12}$$

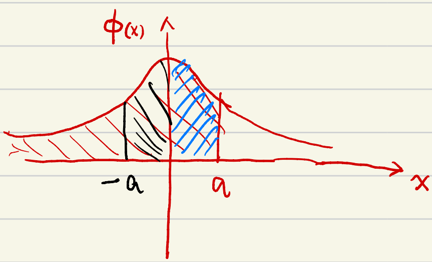
Hence $P\left\{29.5 \le X_1+\cdots+X_{10} \le 40.5\right\}$

$$= P\left\{\frac{29.5-10\times\frac{7}{2}}{\sqrt{350/12}} \le \frac{X_1+\cdots+X_{10}-10\times\frac{7}{2}}{\sqrt{10}\cdot\sqrt{\frac{35}{12}}} \le \frac{40.5-10\times\frac{7}{2}}{\sqrt{350/12}}\right\}$$

$$\approx P\left\{-1.018 \le Z \le 1.018\right\}$$

$$= 2\cdot\Phi(1.018) - 1$$

$$\approx 0.692.$$



Thm 2 (The strong law of large numbers).

Let $X_1, \cdots, X_n, \cdots,$ be an i.i.d. sequence of r.v.'s with a finite mean μ. Then with prob. 1,

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow \mu \qquad \text{as } n \to \infty.$$

In other word,

$$P\left\{ \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right\} = 1.$$

The proof of Thm 2 is based on two Lemmas.

Lem 3. Assume X is a non-negative r.v. which may take the value $+\infty$. Suppose $E[X] < \infty$.

Then $P\{X < \infty\} = 1$.

Pf. For any positive integer n, by the Markov inequality,

$$P\{X = \infty\} \le P\{X \ge n\} \le \frac{E[X]}{n}.$$

Letting $n \to \infty$ gives $P\{X = \infty\} = 0$.

Hence $P\{X < \infty\} = 1 - P\{X = \infty\} = 1$. ☑

Lem 4. Let X be a r.v. Then

$$E[X^4] \geq (E[X^2])^2.$$

Pf. Notice that

$$\mathrm{Var}(X^2) = E[X^4] - (E[X^2])^2.$$

Since $\mathrm{Var}(X^2) \geq 0$, it follows that $E[X^4] \geq (E[X^2])^2$. ▨

Pf of Thm 2.

We will the thm under an additional assumption

$$E[X_i^4] = K < \infty.$$

WLOG, assume $\mu = 0$.

Write $S_n = X_1 + \cdots + X_n$.

We will estimate

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4].$$

Expand $(X_1 + \cdots + X_n)^4$ in terms of

$$X_i^4, \quad X_i^3 X_j, \quad X_i^2 X_j^2, \quad X_i^2 X_j X_k, \quad X_i X_j X_k X_\ell$$

with distinct i, j, k, ℓ.

Notice that $E[X_i^3 X_j] = E[X_i^3] E[X_j] = 0$.

$$E[X_i^2 X_j X_k] = E[X_i^2] E[X_j] E[X_k] = 0.$$

$$E[X_i X_j X_k X_\ell] = 0.$$

Hence

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4]$$

$$= n E[X_i^4] + \binom{n}{2}\binom{4}{2} E[X_i^2 X_j^2]$$

$$= n E[X_i^4] + 6\binom{n}{2} E[X_i^2] E[X_i^2]$$

$$\leq n E[X_i^4] + 6\binom{n}{2} E[X_i^4]$$

(using Lem 4)

$$= (3n^2 - 2n) K$$

$$\leq 3n^2 K, \quad \text{where } K = E[X_i^4] < \infty.$$

So $$E\left[\frac{S_n^4}{n^4}\right] \leq \frac{3K}{n^2}.$$

Hence
$$\sum_{n=1}^{\infty} E\left[\left(\frac{S_n}{n}\right)^4\right] \leq \sum_{n=1}^{\infty} \frac{3K}{n^2} < \infty.$$

Thus
$$E\left[\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4\right] = \sum_{n=1}^{\infty} E\left[\left(\frac{S_n}{n}\right)^4\right] < \infty.$$

Let $X = \sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4$. Then X is a r.v, non-negative (may take the value ∞)

However $E[X] < \infty$

By Lem 3, $P\{X < \infty\} = 1.$

Hence $P\left\{\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4 < \infty\right\} = 1$

However
$$\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4 < \infty \Rightarrow \frac{S_n}{n} \to 0 \text{ as } n \to \infty.$$

Hence $$P\left\{\lim_{n\to\infty} \frac{S_n}{n} = 0\right\} = 1$$

Thus with Prob. 1,

$$\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \to 0 .$$

If $\mu \neq 0$, then letting $\widetilde{X}_n = X_n - \mu$

applying the SLLN to $(\widetilde{X}_n)$ gives

$$\frac{\widetilde{X}_1 + \cdots + \widetilde{X}_n}{n} \to 0 \qquad \text{almost sure.}$$

$$\Longleftrightarrow \quad \frac{X_1 + \cdots + X_n}{n} \to \mu \qquad \text{almost sure.}$$

$\square$