Math 3280 A

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Review:

- Markov inequality: For a r.v. $X \geq 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a} \quad \text{for all } a > 0.$$

- Chebyshev inequality: For a r.v. X with mean μ,

$$P\{|X-\mu| > \varepsilon\} \leq \frac{Var(X)}{\varepsilon^2} \quad \text{for all } \varepsilon > 0.$$

- (The weak law of large numbers)

Let $X_1, X_2, \cdots, X_n, \cdots$ be an i.i.d sequence of r.v's, having a finite mean μ. Then for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \text{ as } n \to \infty.$$

Thm 1. (The central limit Thm).

Let $X_1, \cdots, X_n, \cdots,$ be an i.i.d. sequence of r.v.'s, each having finite mean μ and variance σ^2.

Then $\forall\, a \in \mathbb{R}$,

$$P\left\{ \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma} \leq a \right\} \longrightarrow \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}} dx,$$

as $n \to \infty$.

Remark: Letting $Z_n = \dfrac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma}$, the CLT states that

$$F_{Z_n}(a) \to F_Z(a) \quad \text{for all } a \in \mathbb{R}$$

where Z stands for a standard normal r.v.

To prove the CLT, we state a result without proof.

<u>Lem 1</u>. Let $Z_1, \cdots, Z_n, \cdots$ be a sequence of r.v's with distribution functions F_{Z_n}. Let Z be a r.v. with distribution function F_Z.

Suppose $M_{Z_n}(t) \to M_Z(t)$ for all $t \in \mathbb{R}$ as $n \to \infty$. (Recall $M_Z(t) := E[e^{tZ}]$)

Then

$$F_{Z_n}(t) \to F_Z(t) \quad \text{for each } t \text{ at}$$

which F_Z is cts, as $n \to \infty$.

Pf of the CLT.

First assume $\mu = 0$, $\sigma^2 = 1$.

Let

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}, \quad n = 1, 2, \cdots.$$

Let Z be the standard normal r.v.

Recall $M_Z(t) = e^{t^2/2}$, $t \in \mathbb{R}$.

Hence we only need to prove for $t \in \mathbb{R}$,

① $M_{Z_n}(t) \longrightarrow e^{t^2/2}$ as $n \to \infty$.

Notice that

$$M_{Z_n}(t) = E[e^{t \cdot \frac{X_1 + \cdots + X_n}{\sqrt{n}}}]$$

$$= \prod_{j=1}^{n} E[e^{t X_i/\sqrt{n}}]$$

$$= \left(M_X\left(\frac{t}{\sqrt{n}}\right)\right)^n, \quad \text{where } X = X_1$$

To show ①, it is equivalent to show

② $n \log M_X\left(\frac{t}{\sqrt{n}}\right) \to t^2/2$ as $n \to \infty$.

For convenience, we write

$$L(t) = \log M_X(t).$$

Notice that

$$L'(t) = \frac{M_X'(t)}{M_X(t)}, \quad L''(t) = \frac{M_X''(t) M_X(t) - (M_X'(t))^2}{M_X(t)^2}$$

In particular

$$L'(0) = \frac{M_X'(0)}{M_X(0)} = \frac{E[X]}{1} = \mu = 0$$

$$L''(0) = \frac{M_X''(0) \cdot M_X(0) - M_X'(0)^2}{M_X(0)^2} = \frac{E[X^2]}{1}$$

$$= Var(X) + E[X]^2$$

$$= 1$$

Hence

$$\lim_{n\to\infty} n L\left(\frac{t}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{L\left(\frac{t}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)^2}$$

$$\underline{\underline{\text{Letting } x = \tfrac{1}{\sqrt{n}}}} \lim_{x\to 0} \frac{L(tx)}{x^2}$$

$$\underline{\underline{\text{L'Hopital's rule}}} \lim_{x\to 0} \frac{L'(tx)\cdot t}{2x}$$

$$= \lim_{x\to 0} \frac{L''(tx)\, t^2}{2}$$

$$= \frac{t^2}{2} L''(0) = t^2/2.$$

In the general case,

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n} \cdot \sigma} = \frac{\frac{X_1 - \mu}{\sigma} + \cdots + \frac{X_n - \mu}{\sigma}}{\sqrt{n}}$$

Notice that $\widetilde{X_i} = \frac{X_i - \mu}{\sigma}$ has mean 0 and variance 1

Since $\widetilde{X_1}, \cdots, \widetilde{X_n}, \cdots,$ are i.i.d with mean 0 and variance 1, the distribution of $\frac{\widetilde{X_1} + \cdots + \widetilde{X_n}}{\sqrt{n}}$ converges to the standard normal distribution. $\square$